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CATEGORICAL CONES AND QUADRATIC HOMOLOGICAL PROJECTIVE DUALITY

BY Alexander KUZNETSOV AND Alexander PERRY

ABSTRACT. – We introduce the notion of a categorical cone, which provides a categorification of the classical cone over a projective variety, and use our work on categorical joins to prove that the homologically projectively dual category of a categorical cone is equivalent to a categorical cone of the homologically projectively dual category. We check that the categorical cone construction provides well-behaved categorical resolutions of singular quadrics, which we use to obtain an explicit quadratic version of the main theorem of homological projective duality. As applications, we prove the duality conjecture for Gushel-Mukai varieties, and produce interesting examples of conifold transitions between noncommutative and honest Calabi-Yau threefolds.

RÉSUMÉ. – Nous introduisons la notion de cône catégorique, qui fournit une catégorification du cône classique au-dessus d'une variété projective, et nous utilisons notre travail sur les joints catégoriques pour prouver que le dual projectif homologique d'un cône catégorique est équivalent au cône catégorique de la catégorie duale projective homologique. Nous vérifions que la construction du cône catégorique fournit des résolutions catégoriques qui se comportent bien de quadriques singulières, que nous utilisons pour obtenir une version quadratique explicite du théorème principal de la dualité projective homologique. Comme applications, nous prouvons la conjecture de dualité pour les variétés de Gushel-Mukai, et produisons des exemples intéressants de transitions conifoldes entre des variétés de Calabi-Yau noncommutatives et de vraies variétés de Calabi-Yau de dimension trois.

1. Introduction

This paper is a sequel to [25], where we introduced categorical joins in the context of homological projective duality (HPD). Building on that work, our goals here are to study a categorical version of the classical cone over a projective variety, to use categorical quadratic cones to give a powerful method for studying derived categories of quadratic sections of varieties, and to give several applications.

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1.1. Background

The basic object of HPD is a *Lefschetz variety*, which consists of a variety mapping to a projective space $X \to \mathbf{P}(V)$ equipped with a *Lefschetz decomposition* of its derived category (a special type of semiorthogonal decomposition). The theory in this form was introduced and developed in [12]. At that point it was already clear that the theory is more categorical in nature, and that for applications it is useful to replace the (perfect) derived category $\mathrm{Perf}(X)$ of X by a more general (suitably enhanced) triangulated category $\mathcal{A}$ equipped with a Lefschetz decomposition; the structure of a map $X \to \mathbf{P}(V)$ is then replaced by a $\mathbf{P}(V)$-linear structure (an action of the monoidal category $\mathrm{Perf}(\mathbf{P}(V))$) on $\mathcal{A}$. We call such data a *Lefschetz category* over $\mathbf{P}(V)$ and think of it as of a *noncommutative* Lefschetz variety. The reader is encouraged to focus on the case where $X \to \mathbf{P}(V)$ is an ordinary morphism of varieties for this introduction, and to consult [32, 25] for more details on the noncommutative situation.

The HPD of a (noncommutative) Lefschetz variety $X \to \mathbf{P}(V)$ is another (noncommutative) Lefschetz variety

$$X^{\natural} \to \mathbf{P}(V^{\vee})$$

over the dual projective space, which governs the derived categories of linear sections of X and can be thought of as a categorical version of the classical projective dual. For details and applications of this theory, see [12, 32, 17, 35].

In [25] given a pair of (noncommutative) Lefschetz varieties $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$, we constructed a (noncommutative) Lefschetz variety

$$\mathcal{J}(X_1, X_2) \to \mathbf{P}(V_1 \oplus V_2),$$

called their *categorical join*, which can be thought of as a noncommutative resolution of singularities of the classical join of X_1 and X_2. Moreover, we proved that various classical properties of joins can be lifted to this level; in particular, (under suitable assumptions) the main result of [25] states that there is an equivalence of Lefschetz varieties

$$\mathcal{J}(X_1, X_2)^{\natural} \simeq \mathcal{J}(X_1^{\natural}, X_2^{\natural}) \tag{1.1}$$

over $\mathbf{P}(V_1^{\vee} \oplus V_2^{\vee})$, i.e., the HPD of a categorical join is the categorical join of the HPDs. This leads to numerous applications, including a nonlinear HPD theorem (see also [10]) giving an equivalence between the "essential parts" of the derived categories of the fiber products

$$X_1 \times_{\mathbf{P}(V)} X_2 \qquad \text{and} \qquad X_1^{\natural} \times_{\mathbf{P}(V^{\vee})} X_2^{\natural}.$$

The simplest case of this result—when X_2 is a linear subspace of $\mathbf{P}(V)$ and hence $X_2^{\natural}$ is its orthogonal linear subspace of $\mathbf{P}(V^{\vee})$—reduces to the main theorem of HPD, and other examples of HPD pairs $(X_2, X_2^{\natural})$ provide extensions of this theorem. Such extensions are most useful in cases when both X_2 and $X_2^{\natural}$ have a nice geometric description. One of the goals of this paper is to produce such pairs where both X_2 and $X_2^{\natural}$ are *categorical resolutions* of singular quadrics and to relate in this way quadratic sections of X_1 and $X_1^{\natural}$. Allowing the quadrics to be singular is crucial for applications, as we will explain below in §1.4.

1.2. Categorical cones

Assume given an exact sequence of vector spaces

$$0 \to V_0 \to V \to \bar{V} \to 0 \tag{1.2}$$

and a closed subvariety X of $\mathbf{P}(\bar{V})$. Recall that the *classical cone* over X with vertex $\mathbf{P}(V_0)$ is the strict transform

$$\mathbf{C}_{V_0}(X) \subset \mathbf{P}(V)$$

of X under the linear projection $\mathbf{P}(V) \dashrightarrow \mathbf{P}(\bar{V})$ from $\mathbf{P}(V_0)$. Note that $\mathbf{C}_{V_0}(X)$ is usually highly singular along its vertex $\mathbf{P}(V_0) \subset \mathbf{C}_{V_0}(X)$.

In this paper, given a (noncommutative) Lefschetz variety $X \to \mathbf{P}(\bar{V})$, we construct a (noncommutative) Lefschetz variety

$$\mathcal{C}_{V_0}(X) \to \mathbf{P}(V)$$

called the *categorical cone* which provides (if X is smooth) a categorical resolution of $\mathbf{C}_{V_0}(X)$. The basic idea of the construction is to first replace the classical cone with the *resolved cone* $\tilde{\mathbf{C}}_{V_0}(X) \to \mathbf{P}(V)$ given by the blowup along $\mathbf{P}(V_0) \subset \mathbf{C}_{V_0}(X)$; the resolved cone is the projectivization of the pullback to X of a natural vector bundle on $\mathbf{P}(\bar{V})$, and hence makes sense even when $X \to \mathbf{P}(\bar{V})$ is not an embedding. The categorical cone is then defined as a certain triangulated subcategory of $\mathrm{Perf}(\tilde{\mathbf{C}}_{V_0}(X))$ following a construction in [14], and can be thought of as a noncommutative birational modification of $\tilde{\mathbf{C}}_{V_0}(X)$ along its exceptional divisor.

As we will show, the categorical cone has several advantages over its classical counterpart:

— $\mathcal{C}_{V_0}(X)$ naturally has the structure of a Lefschetz variety over $\mathbf{P}(V)$ induced by that of X (Theorem 3.21).

— $\mathcal{C}_{V_0}(X)$ is smooth and proper if X is (Lemma 3.11).

— $\mathcal{C}_{V_0}(X)$ is defined when $X \to \mathbf{P}(\bar{V})$ is not an embedding, and even when X is noncommutative (Definition 3.6).

For us, however, the main advantage of the categorical cone is its compatibility with HPD: our first main result is the identification of the HPD of a categorical cone with another categorical cone.

In fact, we work in a more general setup than above, that simultaneously allows for extensions of the ambient projective space, because this extra generality is useful in applications (see §1.4). Namely, let V be a vector space and assume given a pair of subspaces

$$V_0 \subset V \qquad \text{and} \qquad V_\infty \subset V^\vee$$

such that $V_0 \subset V_\infty^\perp$, or equivalently $V_\infty \subset V_0^\perp$, where the orthogonals are taken with respect to the natural pairing between V and $V^\vee$. Let

$$\bar{V} = V_\infty^\perp / V_0, \qquad \text{so that} \qquad \bar{V}^\vee \cong V_0^\perp / V_\infty.$$

For $V_\infty = 0$ this reduces to the situation (1.2) above. Let $X \to \mathbf{P}(\bar{V})$ be a Lefschetz variety, with HPD variety $X^\natural \to \mathbf{P}(\bar{V}^\vee)$. The categorical cone $\mathcal{C}_{V_0}(X)$ is then a Lefschetz variety over $\mathbf{P}(V_\infty^\perp)$. Via the inclusion $\mathbf{P}(V_\infty^\perp) \to \mathbf{P}(V)$ we can regard $\mathcal{C}_{V_0}(X)$ as a Lefschetz variety over $\mathbf{P}(V)$, which we write as $\mathcal{C}_{V_0}(X)/\mathbf{P}(V)$ for emphasis. Similarly, we have a Lefschetz variety $\mathcal{C}_{V_\infty}(X^\natural)/\mathbf{P}(V^\vee)$ over $\mathbf{P}(V^\vee)$.