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SPECIALIZATION OF NÉRON-SEVERI GROUPS IN POSITIVE CHARACTERISTIC

BY Emiliano AMBROSI

Abstract. – Let k be an infinite finitely generated field of characteristic $p > 0$. Fix a separated scheme X smooth, geometrically connected, and of finite type over k and a smooth proper morphism $f : Y \to X$. The main result of this paper is that there are "lots of" closed points $x \in X$ such that the fiber of f at x has the same geometric Picard rank as the generic fiber. If X is a curve we show, under a minimal technical assumption, that this is true for all but finitely many k-rational points. In characteristic zero, these results have been proved by André (existence) and Cadoret-Tamagawa (finiteness) using Hodge theoretic methods. To extend the argument in positive characteristic we use the variational Tate conjecture in crystalline cohomology, the comparison between various p-adic cohomology theories and independence techniques. The result has applications to the Tate conjecture for divisors, uniform boundedness of Brauer groups, proper families of projective varieties and to the study of families of hyperplane sections of smooth projective varieties.

Résumé. – Soit k un corps infini de type fini et de caractéristique $p > 0$. Soit X un schéma séparé, lisse, géométriquement connexe et de type fini sur k et $f : Y \to X$ un morphisme propre et lisse. Le résultat principal de cet article est qu'il y a « beaucoup » des points fermés $x \in X$ tels que le rang du groupe de Néron-Severi géométrique de la fibre en x est le même que celui de la fibre générique. Si X est une courbe, sous une hypothèse technique minimale, on montre que cela est vrai pour tous les points k-rationnels sauf un nombre fini. Pour k de caractéristique 0, ces résultats sont dus à André (existence) et Cadoret-Tamagawa (finitude) en utilisant la théorie de Hodge. Pour étendre l'argument en caractéristique positive, on utilise la conjecture variationnelle de Tate en cohomologie cristalline, la comparaison entre différentes cohomologies p-adiques et des techniques d'indépendance. Ce résultat donne lieu à des applications à la conjecture de Tate pour les diviseurs, à la borne uniforme des groupes de Brauer, aux familles propres de variétés projectives et à l'étude des familles de sections hyperplans de variétés projectives lisses.

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1. Introduction

1.1. Conventions

For a field k, a k-variety is a reduced scheme separated and of finite type over k. For a k-variety X, write $|X|$ for the set of closed points. If $x \in X$, write $k(x)$ for its residue field and $\overline{x}$ for a geometric point over x. If $Y \to X$ is a morphism and $x \in X$ write $i_x : Y_x \to Y$ for the natural inclusion of the fiber Y_x at x in Y. We use $\twoheadrightarrow$ and $\hookrightarrow$ to denote surjective and injective maps respectively. If $\mathbb{F}_q$ is a finite field, write $\mathbb{F}$ for its algebraic closure. If $\mathcal{C}$ is an abelian category write $\mathcal{C} \otimes \mathbb{Q}$ for its isogeny category and $\otimes\mathbb{Q} : \mathcal{C} \to \mathcal{C} \otimes \mathbb{Q}$ for the canonical functor.

1.2. Summary

Let k be a finitely generated field of characteristic $p > 0$, $\ell \neq p$ a prime, X a smooth geometrically connected k-variety, and $f : Y \to X$ a smooth proper morphism. On a first approximation, the main result of this paper is a version of the variational Tate conjecture for divisors in the generic case: for $x \in |X|$, if $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ has no more Galois invariants than the generic fiber, then $Y_{\overline{x}}$ has no more divisors than the generic fiber. When k is a field of characteristic zero, this has been proved by André as a consequence of Lefschetz $(1,1)$-theorem and the Hodge theory in [15]; see Section 1.5 for more details.

The starting point of our proof is to replace Hodge theory with crystalline cohomology, since a variational form of the Tate conjecture (Fact 1.6.1.1) is known in this setting. The main difficulty to overcome is to transfer the information about the Galois invariants of the ℓ-adic lisse sheaf $R^2 f_* \mathbb{Q}_\ell(1)$ to the crystalline local system (F-isocrystal) $R^2 f_{\mathrm{crys},*}\mathcal{O}_{Y/K}(1)$. This is the main new contribution of this paper (Theorem 1.6.3.1). More precisely, since the F-isocrystal $R^2 f_{\mathrm{crys},*}\mathcal{O}_{Y/K}(1)$ has a behavior which is quite different from $R^2 f_* \mathbb{Q}_\ell(1)$ (for example, in general its cohomology is not finite dimensional), this comparison cannot be done directly. The idea is then to show (Theorem 6.5.4.1) that $R^2 f_{\mathrm{crys},*}\mathcal{O}_{Y/K}(1)$ is coming from a smaller and better behaved category of p-adic local systems: the category of overconvergent F-isocrystals. As it has been understood that overconvergent F-isocrystals share many properties with lisse sheaves ([14], [27], [4]), the idea is to compare first $R^2 f_{\mathrm{crys},*}\mathcal{O}_{Y/K}(1)$ with its overconvergent incarnation $R^2 f_* \mathcal{O}^\dagger_{Y/K}(1)$ via various p-adic comparison theorems and then $R^2 f_* \mathcal{O}^\dagger_{Y/K}(1)$ with $R^2 f_* \mathbb{Q}_\ell(1)$ via the theory of weights ([17], [25]).

However, the theory of weights allows us to transfer only information readable on characteristic polynomials of the Frobenii, that is to compare $R^2 f_* \mathcal{O}^\dagger_{Y/K}(1)$ and $R^2 f_* \mathbb{Q}_\ell(1)$ only up to semi-simplification. The way to grasp the missing information is to use Tannakian techniques: instead of considering only $R^2 f_* \mathbb{Q}_\ell(1)$, we consider all the possible tensor constructions and sub quotients arising from them, obtaining an algebraic group G_ℓ. Since G_ℓ identifies with the Zariski closure of the image of $\pi_1(X, \overline{x})$ acting on $(R^2 f_* \mathbb{Q}_\ell(1))_{\overline{x}} \simeq H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$, instead of asking that $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ has no more Galois invariants than the generic fiber, we ask that the Zariski closure $G_{\ell,x}$ of the image of $\pi_1(x, \overline{x})$ acting on $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ identifies with G_ℓ. Then, the theory of weights, combined now with some algebraic group theory, allows us to compare the ℓ-adic and the p-adic worlds.

Behind this, there is the idea that, while $R^2 f_*\mathcal{O}^\dagger_{Y/K}(1)$ and $R^2 f_*\mathbb{Q}_\ell(1)$ should be different incarnations of the same motives, each of them contains some specific feature: $R^2 f_*\mathbb{Q}_\ell(1)$ can be studied via ℓ-adic Lie groups theory, while $R^2 f_*\mathcal{O}^\dagger_{Y/K}(1)$ is an overconvergent incarnation of $R^2 f_{\mathrm{crys},*}\mathcal{O}_{Y/K}(1)$, which, in turn, contains information on the deformations of cycles.

1.3. Galois-generic points

Let k be a field of characteristic $p > 0$ with algebraic closure $\overline{k}$, X a smooth geometrically connected k-variety with generic point η and $f : Y \to X$ a smooth proper morphism of k-varieties. For $x \in X$, fix an étale path from $\overline{x}$ to $\overline{\eta}$. For every $\ell \neq p$, by smooth proper base change $R^2 f_*\mathbb{Q}_\ell(1)$ is a lisse sheaf on X and the choice of the étale path gives equivariant isomorphisms

$$\begin{array}{ccccccc} H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)) & \simeq & R^2 f_*\mathbb{Q}_\ell(1)_{\overline{\eta}} & \simeq & R^2 f_*\mathbb{Q}_\ell(1)_{\overline{x}} & \simeq & H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)) \\ \circlearrowright & & & & & & \circlearrowright \\ \pi_1(X,\overline{\eta}) & \simeq & \pi_1(X,\overline{x}) & \longleftarrow & & & \pi_1(x,\overline{x}). \end{array}$$

Definition 1.3.1. – A point $x \in X$ is ℓ-Galois-generic (resp. strictly ℓ-Galois-generic) for $f : Y \to X$ if the image of $\pi_1(x,\overline{x}) \to \pi_1(X,\overline{\eta}) \to \mathrm{GL}(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$ is open (resp. coincides with) in the image of $\pi_1(X,\overline{\eta}) \to \mathrm{GL}(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$.

By [10, Theorem 1.1] (1), x is ℓ-Galois-generic for one $\ell \neq p$ if and only if x is ℓ-Galois-generic for every $\ell \neq p$. So one simply says that x is Galois-generic for f. This is not true for strictly Galois-generic points, and one says that x is strictly Galois-generic if there exists an $\ell \neq p$ such that x is strictly ℓ-Galois-generic.

1.4. Néron-Severi generic points

1.4.1. *Tate conjecture for divisors*. – The geometric Néron-Severi group $\mathrm{NS}(Z_{\overline{k}})$ of a smooth proper k-variety Z is a finitely generated abelian group such that $\mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q}$ identifies with the image of the cycle class map for ℓ-adic cohomology

$$c_{Z_{\overline{k}}} : \mathrm{Pic}(Z_{\overline{k}}) \otimes \mathbb{Q} \to H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1)).$$

Since $\mathrm{NS}(Z_{\overline{k}})$ is a finitely generated abelian group, $\pi_1(k)$ acts on it through a finite quotient and hence $\mathrm{NS}(Z_{\overline{k}}) \subseteq H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$ is fixed under the action of the connected component G_ℓ^0 of the Zariski closure G_ℓ of the image Π_ℓ of $\pi_1(k)$ acting on $H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$. Recall that the ℓ-adic Tate conjecture for divisors ([41]) predicts the following:

Conjecture 1.4.1.1 ($T(Z,\ell)$). – Let k be a finitely generated field and Z a smooth proper k-variety. Then the map $c_{Z_{\overline{k}}} : \mathrm{Pic}(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{G_\ell^0}$ is surjective.

(1) Recall that finite fields are in particular ℓ-non-Lie semisimple for every ℓ different from the characteristic, so that [10, Theorem 1.1] applies in our setting.