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ON THE ALGEBRAIC COBORDISM RING OF INVOLUTIONS

BY Olivier HAUTION

ABSTRACT. – We consider the cobordism ring of involutions of a field of characteristic not two, whose elements are formal differences of classes of smooth projective varieties equipped with an involution, and relations arise from equivariant K-theory characteristic numbers. We investigate in detail the structure of this ring. Concrete applications are provided concerning involutions of varieties, relating the geometry of the ambient variety to that of the fixed locus, in terms of Chern numbers. In particular, we prove an algebraic analog of Boardman's five halves theorem in topology, of which we provide several generalizations and variations.

RÉSUMÉ. – Cet article concerne l'anneau de cobordisme des involutions d'un corps de caractéristique différente de deux. Les éléments de cet anneau sont les différences formelles de deux classes de variétés projectives lisses équipées d'une involution, les relations étant définies à l'aide des nombres caractéristiques en K-théorie équivariante. Nous étudions en détail la structure de cet anneau. Nous décrivons des applications concrètes à propos des involutions des variétés algébriques, reliant la géométrie de la variété ambiante à celle du lieu fixe, en termes de nombres caractéristiques. Nous établissons en particulier un analogue algébrique du « théorème des cinq moitiés » de Boardman en topologie, dont nous fournissons diverses généralisations.

Introduction

A smooth closed manifold is nonbounding (in the unoriented sense) precisely when at least one of its Stiefel-Whitney numbers (with values modulo two) does not vanish. Conner and Floyd [5, (27.1)] observed in 1964 that the fixed loci of involutions of nonbounding manifolds cannot have arbitrary low dimension (compared to the dimension of the ambient manifold), and wondered how to explicitly compute that lower bound. In 1966, Boardman did so in its remarkable "five halves theorem":

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THEOREM 0.1 ([3]). – *Let X be a smooth closed n-manifold equipped with a smooth involution. If X is nonbounding, then at least one component of the fixed locus has dimension greater than or equal to $2n/5$.*

Boardman's proof [4] does not provide a clear geometric reason for the existence of this lower bound, but instead relies on a fine understanding of the multiplicative structure of the unoriented cobordism ring of involutions. Roughly speaking, Boardman constructs an explicit infinite family of manifolds with involutions, which generates that ring as a polynomial algebra over $\mathbb{F}_2$, after a certain stabilization procedure. This reduces the proof to the verification of the validity of the theorem for the elements of this family, which is immediate.

The starting point of the current paper consists in transcribing Conner-Floyd's and Boardman's ideas to the algebro-geometric world. An important difference is that we insist on working "integrally", while unoriented cobordism is intrinsically 2-torsion: more precisely formal multiplication by 2 is nontrivial (in a universal way) in this paper, while it is trivial in the unoriented setting. This yields new results in a range of fixed loci dimensions not captured by unoriented cobordism. In fact involutions on smooth projective varieties which are nonbounding (in the sense that they possess a nonzero Chern number) may very well have low-dimensional fixed loci, so the picture might seem at first quite different from that in algebraic topology. A moment's reflection however reveals that these differences should dissipate once we consider unitary (instead of unoriented) cobordism in topology. We are not aware of analogs of the results of the current paper in that topological setting, although we believe that a completely parallel theory could be developed.

Voevodsky first introduced algebraic cobordism in [26] using homotopical techniques. Later Levine-Morel provided a more geometric construction [15], which is essentially limited to base fields of characteristic zero. In this paper, we will use a third approach pioneered by Merkurjev [17], which is more elementary, and works in arbitrary characteristic. The basic idea is to *define* the cobordism class of a smooth projective variety in the Lazard ring by its collection of Chern numbers, computed using Chow's theory of cycle classes modulo rational equivalence.

By an involution, we will mean for short a smooth projective variety over a fixed base field of characteristic not two, equipped with an action of the algebraic group μ_2 (which is canonically isomorphic to $\mathbb{Z}/2$). It might initially seem natural to define the cobordism ring of involutions using equivariant analogues of the Chern numbers, with values in the equivariant Chow ring of the point. This yields the wrong theory though, which for instance does not distinguish the different possible involutions of a given finite set. This problem arises essentially because the approximations of the classifying space of μ_2 are not cellular varieties. A simple solution consists in using K-theory instead of Chow's theory. The equivariant cobordism ring thus defined does contain the Burnside ring of the group $\mathbb{Z}/2$, and in fact coincides in characteristic zero with the ring obtained using Levine-Morel's algebraic cobordism theory instead of K-theory. These two facts provide a conceptual justification to our definition of the equivariant cobordism ring (while a more concrete justification is provided by the applications obtained in §8). These points are discussed in detail in §2, where more generally cyclic group actions are considered (results of Bix and tom Dieck [2] in topology suggest that this K-theoretic approach should fail for all noncyclic groups).

All elements are "geometric" in nonequivariant cobordism, in the sense that the cobordism ring of the point is generated by classes of smooth projective varieties. This is not true anymore in the equivariant setting. In this paper we study the "geometric" subring $\mathcal{O}(\mu_2)$ inside the "cohomological" cobordism ring $\mathrm{H}_{\mu_2}(k)$ obtained using Borel's construction. The elements of $\mathcal{O}(\mu_2)$ are the classes of virtual involutions, that is, formal differences of (smooth projective k-varieties equipped with) involutions.

The structure of $\mathcal{O}(\mu_2)$ is described in §7 by means of a morphism $\nu\colon \mathcal{O}(\mu_2) \to \mathcal{M}$ mapping the class of an involution to the cobordism class of the normal bundle to its fixed locus (here $\mathcal{M}$ is a polynomial ring over the Lazard ring in variables indexed by natural numbers, and should be thought of as the cobordism ring of BGL). Of course ν vanishes on the classes of involutions without fixed points, but more interestingly such involutions generate the kernel of ν. We also describe the image of ν, providing conditions which permit to decide whether a given vector bundle is cobordant to the normal bundle to the fixed locus of some involution. These results are expressed in the fundamental exact sequence of (7.2.9), which can be compared to Conner-Floyd's sequence in topology [5, (28.1)].

The definition of the topological analog of the morphism ν is quite straight-forward (using the equivariant collaring theorem [5, (21.2)]), and is basic in Conner-Floyd's theory. By contrast, the construction of the morphism ν proved to be a serious problem for us. This is probably inherent to our elementary approach to cobordism, but that particular problem seems unlikely to disappear if one uses Levine-Morel's theory instead, because there are more cobordism relations than just naive cobordisms as in topology (certain degenerations must be allowed [16]). Our solution consists in developing a substantial part of the theory before even constructing the morphism ν. In particular, the multiplicative structures of $\mathcal{O}(\mu_2)$ and $\mathcal{M}$ are used in an essential way, and so is the analog (6.2.10) of the injectivity of Boardman's J-homomorphism, a result appearing only at the very end of Boardman's paper [4, Corollary 17].

As a first step, we introduce in §3 the characteristic class γ (of a vector bundle). Three equivalent constructions are provided: the first uses the formal group law, the second arises from a certain stabilization procedure applied to the associated projective bundle, and the third involves $\mathbb{G}_m$-equivariant considerations. The interplay between the different natures of these approaches can be exploited fruitfully: for instance, the multiplicative property of γ is clear from the first definition, but not at all from the other two. From the class γ we derive the class g, the analog of Boardman J-homomorphism [3]. Boardman's approach is closer to our second construction, which explains that the multiplicativity of Boardman J-homomorphism was a delicate point to establish (according to Boardman [3]: "There ought to be a direct geometric proof that J' is a ring homomorphism").

It is interesting to note that the exact same characteristic class γ appears in the formulation of Quillen's formula, expressing the class of a projective bundle in the cohomology of its base, and thus bearing no apparent relation with involutions. This formula was first stated by Quillen for complex cobordism [21, Theorem 1], then by Panin-Smirnov [20] for oriented cohomology theories, and Vishik gave a complete proof for algebraic cobordism in characteristic zero [25, §5.7]. As a byproduct, we derive a new proof of that formula in (3.2.4),