ISSN 0012-9593

quatrième série - tome 56 fascicule 5 septembre-octobre 2023

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 1 048). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 56, 2023, p. 1313 à 1337

ON THE COLLAPSING OF HOMOGENEOUS BUNDLES IN ARBITRARY CHARACTERISTIC

BY ANDRÁS CRISTIAN LŐRINCZ

ABSTRACT. – We study the geometry of equivariant, proper maps from homogeneous bundles $G \times_P V$ over flag varieties G/P to representations of G, called collapsing maps. Kempf showed that, provided the bundle is completely reducible, the image $G \cdot V$ of a collapsing map has rational singularities in characteristic zero. We extend this result to positive characteristic and show that for the analogous bundles the saturation $G \cdot V$ is strongly F-regular if its coordinate ring has a good filtration. We further show that in this case the images of collapsing maps of homogeneous bundles restricted to Schubert varieties are F-rational in positive characteristic, and have rational singularities in characteristic zero. We provide results on the singularities and defining equations of saturations $G \cdot X$ for P-stable closed subvarieties $X \subset V$. We give criteria for the existence of good filtrations for the coordinate ring of $G \cdot X$.

Our results give a uniform, characteristic-free approach for the study of the geometry of a number of important varieties: multicones over Schubert varieties, determinantal varieties in the space of matrices, symmetric matrices, skew-symmetric matrices, and certain matrix Schubert varieties therein, representation varieties of radical square zero algebras (e.g., varieties of complexes), subspace varieties, higher rank varieties, etc.

RÉSUMÉ. – On étudie la géométrie des applications propres équivariantes de fibrés homogènes $G \times_P V$ sur les variétés de drapeaux G/P dans les représentations de G, appelées applications d'effondrement. Kempf a montré que lorsque le fibré est complètement réductible, l'image $G \cdot V$ d'une application d'effondrement a des singularités rationnelles en caractéristique zéro. On étend ce résultat à la caractéristique positive et on montre que pour les fibrés analogues la saturation $G \cdot V$ est fortement F-régulière si son anneau des coordonnées a une bonne filtration. De plus, on montre que dans ce cas les images des applications d'effondrement de fibrés homogènes restreintes aux variétés de Schubert sont F-rationnelles en caractéristique positive, et ont des singularités rationnelles en caractéristique zéro. On obtient des résultats sur les singularités et les équations qui définissent les saturations $G \cdot X$ pour les sous-variétés $X \subset V$ fermées P-stables. On donne un critère pour l'existence de bonnes filtrations pour l'anneau des coordonnées de $G \cdot X$.

Nos résultats fournissent une approche uniforme et indépendante de la caractéristique, à l'étude de la géométrie de nombreuses variétés importantes: multicônes sur les variétés de Schubert, variétés déterminantales dans l'espace de matrices, matrices symétriques, matrices antisymétriques et certaines variétés de Schubert de matrices, variétés de représentations des algèbres dont le carré du radical est zéro (par ex. variétés de complexes), variétés de sous-espaces, variétés de rang supérieur, etc.

doi:10.24033/asens.2556

1. Introduction

Let G be a connected reductive group over an algebraically closed field $\Bbbk$. Consider a parabolic subgroup P of G, and let W be a G-module and $V \subset W$ a P-stable submodule. The saturation $G \cdot V \subset W$ is the image of the homogeneous vector bundle $G \times_P V$ under the proper "collapsing map" $G \times_P V \to W$ induced by the action of G on W.

Many remarkable varieties can be realized through such collapsing of bundles for various choices of G, P, W, V (cf. Section 4; for more such examples, see [67]). Generally, the study of their geometry has been undertaken on case-by-case basis. An exception is the seminal work [40], where it is shown that in characteristic zero $G \cdot V$ has rational singularities whenever the unipotent radical $U(P)$ of P acts trivially on V (see also [41]). Further, in this case the singularities of $G \cdot X$ are shown to be well-behaved for a closed P-stable subvariety $X \subset V$ [40, Proposition 1 and Theorem 3].

In this paper, we generalize and extend the scope of Kempf's results along several directions. In particular, we give characteristic-free strengthenings of the statements above, under the presence of good filtrations as initiated by Donkin [14, 16]. We say that a G-variety Z is good, if $\Bbbk[Z]$ has a good filtration (see Section 2.4). We point out to the reader that all good-related properties hold automatically when $\operatorname{char} \Bbbk = 0$, and our results below are new in this case as well (with the exception of Theorem 1.3).

Let $B \subset P$ a Borel subgroup of G and $T \subset B$ a maximal torus. We denote the set of dominant weights of G by $X(T)_+$. For $\lambda \in X(T)_+$ we let $\Delta_G(\lambda)$ denote the corresponding Weyl module (see Section 2.2). We consider the Levi decomposition $P = L \ltimes U(P)$ with L reductive. Pick any $\lambda_1, \lambda_2, \ldots, \lambda_n \in X(T)_+$, and for the rest of the introduction fix

$$W = \bigoplus_{i=1}^{n} \Delta_G(\lambda_i) \quad \text{and} \quad V = \bigoplus_{i=1}^{n} \Delta_L(\lambda_i). \tag{1.1}$$

We have a natural inclusion $V \subseteq W^{U(P)}$, with equality if $\operatorname{char} \Bbbk = 0$ (when the bundle is completely reducible [40]). While the examples in Section 4 fit into the setup (1.1), we note that in Section 3 we develop the results in a more general setting (see (3.1)).

THEOREM 1.2. – *Let $X \subset V$ be an L-submodule such that $G \cdot X$ is good. Then $G \cdot X$ is strongly F-regular when* $\operatorname{char} \Bbbk > 0$ *(resp. is of strongly F-regular type when* $\operatorname{char} \Bbbk = 0$*).*

This illustrates that good filtrations are responsible for the geometric behavior of saturations in positive characteristic, a phenomenon that is apparent in invariant theory as well [27, 31]. Example 4.4 demonstrates that this assumption cannot be dropped.

The following is our main criterion for the existence of good filtrations (for the definition of good pairs, see Section 2.4).

THEOREM 1.3. – *Assume that W is good, and that (V, X) is a good pair for some closed L-variety $X \subset V$. Then $(W, G \cdot X)$ is a good pair of G-varieties.*

In particular, this implies that $G \cdot V$ is good whenever $\operatorname{char} \Bbbk > \max\{\dim \Delta_G(\lambda_i) \mid 1 \leq i \leq n\}$. However, in concrete situations the bound on $\operatorname{char} \Bbbk$ can be further improved significantly (cf. Sections 4.1, 4.2). See Theorem 3.6 for other criteria in this direction.

We extend the collapsing method to various relative settings, thus greatly increasing its versatility. These include restrictions to Schubert varieties or multiplicity-free subvarieties of flag varieties (for the latter, see Corollary 3.14). Below $\mathcal{W}$ denotes the Weyl group of G.

THEOREM 1.4. – *Consider a closed L-variety $X \subset V$ and assume that $G \cdot X$ is good. For any $w \in \mathcal{W}$, we have:*

1. *$\overline{BwX}$ is normal if and only if X is so.*
2. *If $\operatorname{char} \Bbbk = 0$, then $\overline{BwX}$ has rational singularities if and only if so does X.*
3. *If $\operatorname{char} \Bbbk > 0$ and X is an L-submodule of V, then $\overline{BwX}$ is F-rational.*

Note that when w is the longest element in $\mathcal{W}$, we have $\overline{BwX} = G \cdot X$.

Frequently (e.g., when $G \cdot X$ is a spherical variety), the varieties $\overline{BwX}$ are orbit closures under the action of the Borel subgroup B (see Section 4). The singularities of such varieties have been investigated mostly in the spherical case (e.g., [59, 9, 12]), but they are not well understood [56, Comments 4.4.4]. Theorem 1.4 is one of the first of its kind at this level of generality, applicable equally in non-spherical situations as well.

When P is itself a Borel subgroup, we sharpen some results on singularities (see Corollary 3.13), extending the case of multicones over Schubert varieties [42, 29].

Next, we provide a relative result on the defining ideals of saturations $G \cdot X$. For this, we introduce the notion of good generators of an ideal, see Definition 2.13.

THEOREM 1.5. – *Let (V, X) be a good pair with $G \cdot V$ good, and denote by $I_X \subset \Bbbk[V]$ the defining ideal of $X \subset V$. Let M be the span of a set of good generators of I_X and take a basis $\mathcal{P}'$ of the G-module $H^0(G/P, \mathcal{V}(M)) \subset \Bbbk[G \cdot V]$. Consider:*

1. *A set of generators $\mathcal{P}_{G \cdot V}$ of the defining ideal $I_{G \cdot V} \subset \Bbbk[W]$ of $G \cdot V$;*
2. *A lift $\tilde{\mathcal{P}}' \subset \Bbbk[W]$ of the set $\mathcal{P}' \subset \Bbbk[W]/I_{G \cdot V}$.*

Then the defining ideal of $G \cdot X$ in $\Bbbk[W]$ is generated by the set $\mathcal{P}_{G \cdot V} \cup \tilde{\mathcal{P}}'$.

In Theorem 3.15 we give a version of the above that yields good defining equations, which we use to readily find (good) defining equations for the examples in Sections 4.1 and 4.2.

Saturations of the type $G \cdot V$ appear in various forms throughout the existing literature, and a range of techniques have been developed to better understand their geometry. Applying the results above in the special case of radical square zero algebras (see Section 4.2), we simultaneously sharpen and generalize the main results in [39, 13, 63, 7, 64, 53, 54] that concern the singularities and defining equations of the Buchsbaum-Eisenbud varieties of complexes as well as varieties of complexes of other type. In addition, we obtain that certain B-orbit closures in varieties of complexes are F-rational when $\operatorname{char} \Bbbk > 0$ (resp. have rational singularities when $\operatorname{char} \Bbbk = 0$).

Our results provide a general method for the investigation of the geometry of parabolically induced orbit closures in a representation W of a reductive group G. Namely, for any choice of a parabolic $P \subset G$, we can take the representation V of the smaller reductive group L as in (1.1) with trivial $U(P)$-action; choosing an L-orbit closure $X = \overline{Lx}$ (for any $x \in V$), saturation gives a G-orbit closure $G \cdot X = \overline{Gx} \subset G \cdot V \subset W$. By considering all such possible