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ALMOST COMMUTING MATRICES, COHOMOLOGY, AND DIMENSION

BY DOMINIC ENDERS AND TATIANA SHULMAN

ABSTRACT. – It is an old problem to investigate which relations for families of commuting matrices are stable under small perturbations, or in other words, which commutative C^*-algebras $C(X)$ are matricially semiprojective. Extending the works of Davidson, Eilers-Loring-Pedersen, Lin and Voiculescu on almost commuting matrices, we identify the precise dimensional and cohomological restrictions for finite-dimensional spaces X and thus obtain a complete characterization: $C(X)$ is matricially semiprojective if and only if $\dim(X) \leq 2$ and $H^2(X;\mathbb{Q}) = 0$.

We give several applications to lifting problems for commutative C^*-algebras, in particular to liftings from the Calkin algebra and to ℓ-closed C^*-algebras in the sense of Blackadar.

RÉSUMÉ. – Un vieux probléme consiste á chercher, pour des familles de matrices commutant quelles relations sont stables sous de petites perturbations, ou en d'autres termes, quelles C*-algébres commutatives $C(X)$ sont matriciellement semi-projectives. En prolongeant les travaux de Davidson, Eilers-Loring-Pedersen, Lin et Voiculescu sur les matrices commutant presque, nous identifions les restrictions dimensionnelles et cohomologiques précises pour l'espace de dimension finie X et obtenons ainsi une caractérisation complète: $C(X)$ est matriciellement semi-projectif si et seulement si $\dim(X) \leq 2$ et $H^2(X;\mathbb{Q}) = 0$.

Nous donnons plusieurs applications aux problémes de relèvement pour les C*-algébres commutatives, en particulier aux relèvements de l'algébre de Calkin et aux C*-algébres ℓ-fermés dans le sens de Blackadar.

1. Introduction

Questions concerning almost commuting matrices have been studied for many decades. Originally, these types of questions first appeared in the 1960s and were popularized by Paul Halmos who included one such question in his famous list of open problems [26]. Specifically he asked:

Is it true that for every $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if matrices* A, B *satisfy*

$$\|A\| \leq 1, \|B\| \leq 1 \text{ and } \|AB - BA\| < \delta,$$

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then the distance from the pair A, B to the set of commuting pairs is less than ε? (Here ε should not depend on the size of matrices, that is on the dimension of the underlying space. The norm above is the operator norm.)

Is the same true under the additional assumption that A and B are Hermitian?

The first significant progress was made by Voiculescu [43] who observed that almost commuting unitary matrices need not be close to commuting unitary matrices. Later, in [8], Choi gave an example of almost commuting matrices which were not close to any commuting ones, thus answering the first of Halmos's questions above. Shortly thereafter Exel and Loring found an alternative, short and elementary proof of Voiculescu's result which in addition revealed that Voiculescu's pair was also not close to any pair of commuting matrices [19].

Already in [43], Voiculescu explained that questions about almost commuting matrices have a C*-algebraic nature and relate to a particular type of lifting property. To make this precise, consider the C*-algebra

$$\prod M_n(\mathbb{C}) = \{(T_n)_{n\in\mathbb{N}} \mid T_n \in M_n(\mathbb{C}),\ \sup_n \|T_n\| < \infty\},$$

together with the ideal $\bigoplus M_n(\mathbb{C})$ of sequences $(T_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \|T_n\| = 0$. Then the question of whether matrices almost satisfying some relations $\mathcal{R}$ are always close to matrices exactly satisfying these relations can be reformulated as a lifting problem for the corresponding universal C*-algebra $C^*(\mathcal{R})$ of these relations:

$$\begin{array}{ccc} & & \prod M_n(\mathbb{C}) \\ & \nearrow & \downarrow \\ C^*(\mathcal{R}) & \longrightarrow & \prod M_n(\mathbb{C})/\bigoplus M_n(\mathbb{C}). \end{array}$$

Voiculescu discusses the question of whether $C(X)$ has this property for various compact metrizable spaces X. However, the lifting property as described above makes perfect sense for arbitrary, not necessarily commutative C*-algebras as well and has been given many different names by various authors—matricial stability, matricial weak semiprojectivity, matricial semiprojectivity, just to name a few. We adopt the latter name here.

In this terminology, Voiculescu's result on two almost commuting unitaries states that $C(\mathbb{T}^2)$, the universal C*-algebra for two commuting unitaries, is actually not matricially semiprojective. Similarly, Halmos's question about almost commuting Hermitian matrices now reads as whether $C(X)$ is matricially semiprojective for $X = [0,1]^2$. This problem remained unanswered for a very long time, but was eventually solved by Lin in the positive [31]. Many further results and techniques around matricial semiprojectivity of $C(X)$ have been established over time, and numerous applications to C*-theory have been found. It is impossible to trace and name them all here, so we only mention [4] and [45] as two of the most recent examples to illustrate the point.

Beyond the world of C*-algebras, almost commuting matrices have found their way into many other areas of mathematics. Striking applications can be found in operator theory (see for instance [22, 2, 30]), quantum physics and condensed matter physics (see e.g., [27] and references therein), and even computer science (see e.g., [24] and references therein).

Variations in which matrices almost commute with respect to various norms other than the operator norm have also found applications, e.g., in group theory (see for instance [16, 25, 13, 1] and references therein).

Coming back to the central question of this paper—*For which compact metrizable space X is $C(X)$ matricially semiprojective?*—we summarize the state of the art, i.e., the main examples for which the answer is known, below.

Compact space X	Is $C(X)$ matricially semiprojective?	Reference
$\mathbb{T}^2$ (2-torus)	No	Voiculescu [43], a short proof by Exel and Loring [19]
$[0,1]^2$	Yes	Lin [31], a short proof by Friis and Rørdam [21]
$[0,1]^3$	No	Voiculescu [42], Davidson [10]
S^2 (2-dimensional sphere)	No	Voiculescu ([42] + a remark in [43]), Loring [32]
$\mathbb{R}P^2$ (real projective plane)	Yes	Eilers, Loring, Pedersen [15]
1-dimensional CW-complexes	Yes	Loring [33]

Despite receiving great attention, this problem still remains open. In this paper we solve the problem under the very mild additional assumption that X has finite covering dimension. (In terms of generators and relations, this means we only look at *finite* families of matrices (almost) satisfying possibly infinitely many relations (see Prop. 2.10).)

Before stating our main result, we would like to mention a few other important lifting properties for C*-algebras which are related to matricial semiprojectivity—projectivity, semiprojectivity, and weak semiprojectivity (see Section 2 for definitions). In contrast to matricial semiprojectivity, the question when $C(X)$ has one of these other lifting properties has been successfully resolved [7, 40, 17]. An important first step in the solutions to all these cases was to obtain information on X by restricting to the category of commutative C*-algebras, i.e., by only considering lifting problems with commutative targets. This immediately led to the understanding that $C(X)$ can be projective only when X is an absolute retract, and that $C(X)$ can be (weakly) semiprojective only when X is an (approximate) absolute neighborhood retract. Thus in all three cases the spaces involved were reasonably well-behaved, and this information was fully used later on. We want to point out that nothing like this is possible for matricial semiprojectivity. Restricting to the commutative subcategory gives no extra information in this case (we make this precise in Remark 5.3) and we are thus forced to cope with general spaces, possibly without any form of regularity. This is one of the main points which distinguishes matricial semiprojectivity from these other lifting properties and which makes it highly intractable.

In this paper, it is therefore crucial to develop a technique which allows us to keep track of our lifting property of interest while approximating a space by nicer, more regular ones. Using approximations of compact, metrizable spaces by CW-complexes, we manage to reduce the problem to this more tractable case. Even for this case the question is still open,