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GEOMETRIC GENERATION OF THE WRAPPED FUKAYA CATEGORY OF WEINSTEIN MANIFOLDS AND SECTORS

BY Baptiste CHANTRAINE, Georgios DIMITROGLOU RIZELL, Paolo GHIGGINI AND Roman GOLOVKO

Abstract. – We prove that the wrapped Fukaya category of any $2n$-dimensional Weinstein manifold (or, more generally, Weinstein sector) W is generated by the unstable manifolds of the index n critical points of its Liouville vector field. Our proof is geometric in nature, relying on a surgery formula for Floer cohomology and the fairly simple observation that Floer cohomology vanishes for Lagrangian submanifolds that can be disjoined from the isotropic skeleton of the Weinstein manifold. Note that we do not need any additional assumptions on this skeleton. By applying our generation result to the diagonal in the product $W \times W$, we obtain as a corollary that the open-closed map from the Hochschild homology of the wrapped Fukaya category of W to its symplectic cohomology is an isomorphism, proving a conjecture of Seidel. We work mainly in the "linear setup" for the wrapped Fukaya category, but we also extend the proofs to the "quadratic" and "localisation" setup. This is necessary for dealing with Weinstein sectors and for the applications.

Résumé. – Nous démontrons que la catégorie de Fukaya enroulée d'une variété (ou plus généralement d'un secteur) de Weinstein W de dimension $2n$ est engendrée par les variétés instables des points critiques d'indice n de son champ de Liouville. Notre preuve, de nature géométrique, repose sur une formule pour la cohomologie de Floer d'une chirurgie et sur l'observation relativement simple que la cohomologie de Floer d'une lagrangienne disjointe du squelette isotrope de la variété de Weinstein s'annule (aucune condition supplémentaire n'est demandée au squelette). En appliquant le critère d'engendrement au produit $W \times W$ nous obtenons en corollaire que l'application ouverte-fermée de l'homologie de Hochschild de la catégorie de Fukaya enroulée de W vers sa cohomologie symplectique est un isomorphisme, prouvant une conjecture de Seidel. Nous travaillons principalement avec la définition « linéaire » de la catégorie de Fukaya enroulée mais nous étendons les preuves aux définitions « quadratique » et « par localisation ». Ces modifications sont nécessaires pour traiter les secteurs de Weinstein et pour certaines applications.

1. Introduction

The *wrapped Fukaya category* is an A_∞-category associated to any Liouville manifold. Its objects are exact Lagrangian submanifolds which are either compact or cylindrical at infinity, possibly equipped with extra structure, the morphism spaces are wrapped Floer

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chain complexes, and the A_∞ operations are defined by counting perturbed holomorphic polygons with Lagrangian boundary conditions. Wrapped Floer cohomology was defined by A. Abbondandolo and M. Schwarz [1], at least for cotangent fibers, but the general definition and the chain level construction needed to define an A_∞-category are due to M. Abouzaid and P. Seidel [4]. The definition of the wrapped Fukaya category was further extended to the relative case by Z. Sylvan, who introduced the notions of *stop* and *partially wrapped Fukaya category* in [39], and by S. Ganatra, J. Pardon and V. Shende, who later introduced the similar notion of *Liouville sector* in [23].

In this article we study the wrapped Fukaya category of Weinstein manifolds and sectors. In the absolute case our main result is the following.

THEOREM 1.1. – *If $(W, \theta, \mathfrak{f})$ is a $2n$-dimensional Weinstein manifold of finite type, then its wrapped Fukaya category $\mathcal{WF}(W, \theta)$ is generated by the Lagrangian cocore planes of the index n critical points of $\mathfrak{f}$.*

In the relative case (i.e., for sectors) our main result is the following. We refer to Section 2.3 for the definition of the terminology used in the statement.

THEOREM 1.2. – *The wrapped Fukaya category of the Weinstein sector $(S, \theta, \mathfrak{f})$ is generated by the Lagrangian cocore planes of its completion $(W, \theta_W, \mathfrak{f}_W)$ and by the spreading of the Lagrangian cocore planes of its belt $(F, \theta_F, \mathfrak{f}_F)$.*

REMARK 1.3. – Exact Lagrangian submanifolds are often enriched with some extra structure: Spin structures, grading or local systems. We ignore them for simplicity, but the same arguments carry over also when that extra structure is considered.

Generators of the wrapped Fukaya category are known in many particular cases. We will not try to give a comprehensive overview of the history of this recent but active subject because we would not be able to make justice to everybody who has contributed to it. However, it is important to mention that F. Bourgeois, T. Ekholm and Y. Eliashberg in [9] sketch a proof that the Lagrangian cocore disks split-generate the wrapped Fukaya category of a Weinstein manifold of finite type. Split-generation is a weaker notion than generation, which is sufficient for most applications, but not for all; see for example [28]. Moreover, Bourgeois, Ekholm and Eliashberg's proposed proof relies on their Legendrian surgery formula, whose analytic details are not complete (see [15] for recent development in that direction).

Most generation results so far, including that of Bourgeois, Ekholm and Eliashberg, rely on Abouzaid's split-generation criterion [2]. On the contrary, our proof is more direct and similar in spirit to Seidel's proof in [37] that the Lagrangian thimbles generate the Fukaya-Seidel category of a Lefschetz fibration or to Biran and Cornea's cone decomposition of Arnol'd type Lagrangian cobordisms [7]. Theorems 1.1 and 1.2 have been proved independently also by Ganatra, Pardon and Shende in [24, Theorem 1.10].

A product of Weinstein manifolds is a Weinstein manifold. Therefore, by applying Theorem 1.1 to the diagonal in a twisted product, and using results of S. Ganatra [22] and Y. Gao [25], we obtain the following result.

Corollary 1.4. – *Let* $(W, \theta, \mathfrak{f})$ *be a Weinstein manifold of finite type. Let* $\mathcal{D}$ *be the full* A_∞ *subcategory of* $\mathcal{WF}(W, \theta)$ *whose objects are the Lagrangian cocore planes. Then the open-closed map*

$$\mathcal{OC}\colon\ HH_*(\mathcal{D}, \mathcal{D}) \to SH^*(W) \tag{1}$$

is an isomorphism.

In Equation (1) HH_* denotes Hochschild homology, SH^* denotes symplectic cohomology and $\mathcal{OC}$ is the open-closed map defined in [2]. Corollary 1.4 in particular proves that

$$\mathcal{OC}\colon\ HH_*(\mathcal{WF}(W, \theta), \mathcal{WF}(W, \theta)) \to SH^*(W) \tag{2}$$

is an isomorphism. This proves a conjecture of Seidel in [38] for Weinstein manifolds of finite type. Note that a proof of this conjecture, assuming the Legendrian surgery formula of Bourgeois, Ekholm and Eliashberg was given by S. Ganatra and M. Maydanskiy in the appendix of [9].

The above result implies in particular that Abouzaid's generation criterion [2] is satisfied for the subcategory consisting of the cocore planes of a Weinstein manifold, from which one can conclude that the cocores split-generate the wrapped Fukaya category. In the exact setting under consideration this of course follows a fortiori from Theorem 1.1, but there are extensions of the Fukaya category in which this generation criterion has nontrivial implications. Notably, this is the case for the version of the wrapped Fukaya category for monotone Lagrangians, as we proceed to explain.

The wrapped Fukaya category as well as symplectic cohomology were defined in the monotone symplectic setting in [35] using coefficients in the Novikov field. When this construction is applied to exact Lagrangians in an exact symplectic manifold, a change of variables $x \mapsto t^{-\mathcal{A}(x)}x$, where $\mathcal{A}(x)$ is the action of the generator x and t is the formal Novikov parameter, allows for an identification of the Floer complexes and the open-closed map with the original complexes and map tensored with the Novikov field. The generalization of Abouzaid's generation criterion to the monotone setting established in [35] thus shows that

Corollary 1.5. – *The wrapped Fukaya category of monotone Lagrangian submanifolds of a Weinstein manifold which are unobstructed in the strong sense (i.e., with* $\mu^0 = 0$*, where* μ^0 *is the number of Maslov index two holomorphic disks passing through a generic point) is split-generated by the Lagrangian cocore planes of the Weinstein manifold.*

Remark 1.6. – The strategy employed in the proof of Theorem 1.1 for showing generation fails for non-exact Lagrangian submanifolds in two crucial steps: in Section 7 and Section 8. First, there are well known examples of unobstructed monotone Lagrangian submanifolds in a Weinstein manifold which are Floer homologically nontrivial even if they are disjoint from the skeleton. Second, our treatment of Lagrangian surgeries requires that we lift the Lagrangian submanifolds in W to Legendrian submanifolds of $W \times \mathbb{R}$, and this is possible only for exact Lagrangian submanifolds. It is unclear to us whether it is true that the cocores generate (and not merely split-generate) the $\mu^0 = 0$ part of the monotone wrapped Fukaya category.