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LOCAL MODELS FOR THE TRIANGULINE VARIETY AND PARTIALLY CLASSICAL FAMILIES

BY ZHIXIANG WU

ABSTRACT. – We generalize Breuil-Hellmann-Schraen's local model for the trianguline variety to certain points with non-regular Hodge-Tate weights. With the local models we are able to prove, under the Taylor-Wiles hypothesis, the existence of certain companion points on the global eigenvariety and the appearance of related companion constituents in the completed cohomology for non-regular crystalline Galois representations. The new ingredients in the proof of the global applications are results relating the partial classicality of locally analytic representations (the existence of non-zero locally algebraic vectors in the parabolic Emerton's Jacquet modules), the partially de Rham properties of Galois representations (the de Rhamness of graded pieces along the paraboline filtrations of the associated (φ, Γ)-modules over the Robba rings) and the relevant properties of cycles on the generalized Steinberg varieties. We prove that partial classicality implies partial de Rhamness in finite slope cases using Ding's partial eigenvarieties.

RÉSUMÉ. – Nous généralisons le modèle local de Breuil-Hellmann-Schraen pour la variété trianguline à certains points à poids de Hodge-Tate non régulier. Avec les modèles locaux, nous prouvons, sous l'hypothèse de Taylor-Wiles, l'existence de certains points compagnons sur la variété de Hecke et l'apparition de constituants compagnons correspondants dans la cohomologie complétée pour les représentations galoisiennes cristallines non régulières. Les nouveaux ingrédients dans la preuve des applications globales sont des résultats mettant en relation la classicité partielle des représentations localement analytiques (l'existence de vecteurs localement algébriques non nuls dans les modules de Jacquet-Emerton paraboliques), les propriétés de De Rham partielles des représentations galoisiennes (propriété de de Rham des morceaux gradués des filtrations parabolines des (φ, Γ)-modules associés sur les anneaux de Robba) et les propriétés correspondantes des cycles sur les variétés de Steinberg généralisées. Nous prouvons que la classicité partielle implique les propriétés de de Rham partielles dans les cas de pente finie en utilisant les variétés de Hecke partielles de Ding.

1. Introduction

Let p be a prime number. This paper concerns p-adic automorphic forms of definite unitary groups and the locally analytic aspect of the p-adic local Langlands program. Its

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aim is to generalize several results of Breuil-Hellmann-Schraen in [20] (local model for the trianguline variety, existence of companion points on the eigenvariety, locally analytic socle conjecture, etc.) to the cases where the Hodge-Tate weights are non-regular (i.e., not pairwise distinct).

1.1. Companion points and main results

Let F^+ be a totally real number field and S_p the set of places of F^+ above p. Let F be a quadratic imaginary extension of F^+ such that every place in S_p splits in F, $n \geq 2$ an integer and $\mathbb{G}$ a totally definite unitary group in n variables over F^+ that is split over F. We fix an open compact subgroup $U^p = \prod_{v \nmid p} U_v$ of $\mathbb{G}(\mathbf{A}_{F^+}^{p\infty})$ and a finite extension L of $\mathbb{Q}_p$ with residue field k_L. For all $v \in S_p$, let $\Sigma_v := \{\tau : F_v^+ \hookrightarrow L\}$ and we assume $|\Sigma_v| = [F_v^+ : \mathbb{Q}_p]$. For each $v \in S_p$, we fix a place $\widetilde{v}$ of F above v and identify $F_v^+ \simeq F_{\widetilde{v}}$. The space of p-adic automorphic forms on $\mathbb{G}$ of tame level U^p, denoted by $\widehat{S}(U^p, L)$, consists of continuous functions $\mathbb{G}(F^+) \setminus \mathbb{G}(\mathbf{A}_{F^+}^{\infty})/U^p \to L$. Let $G_p = \prod_{v \in S_p} G_v$ be the p-adic Lie group $\mathbb{G}(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) = \prod_{v \in S_p} \mathbb{G}(F_v^+)$. Let $B_p = \prod_{v \in S_p} B_v$ (resp. $T_p = \prod_{v \in S_p} T_v$) be the Borel subgroup (resp. the maximal torus) of $G_p \simeq \prod_{v \in S_p} \mathrm{GL}_n(F_v^+)$ consisting of upper-triangular (resp. diagonal) matrices. Then G_p acts on $\widehat{S}(U^p, L)$ via right translations. We assume furthermore that $p > 2$ and $\mathbb{G}$ is quasi-split at all finite places of F^+. Let $\overline{F}$ be an algebraic closure of F. We fix a (modular) absolutely irreducible Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(k_L)$ so that $\overline{\rho}$ is associated with a maximal ideal of some usual Hecke algebra acting on $\widehat{S}(U^p, L)$.

After Emerton [38], one way to construct eigenvarieties, rigid analytic varieties parameterizing finite slope overconvergent p-adic eigenforms, is using Emerton's Jacquet module functor for locally analytic representations of p-adic Lie groups. There exists a rigid space over L (our eigenvariety), denoted by $Y(U^p, \overline{\rho})$, on which a point is a pair $(\rho, \underline{\delta})$, where ρ is a p-adic continuous n-dimensional representation of $\mathrm{Gal}(\overline{F}/F)$ and $\underline{\delta}$ is a continuous character of T_p which appears in $J_{B_p}(\Pi(\rho)^{\mathrm{an}})$. Here $\Pi(\rho)$ is the sub-G_p-representation of $\widehat{S}(U^p, L)$ associated with ρ cut out by a prime ideal of certain Hecke algebra, $\Pi(\rho)^{\mathrm{an}}$ is the subspace of $\Pi(\rho)$ consisting of locally analytic vectors which is an admissible locally analytic representation of G_p and $J_{B_p}(-)$ denotes the Emerton's Jacquet module functor so that $J_{B_p}(\Pi(\rho)^{\mathrm{an}})$ is a locally analytic representation of the Levi subgroup T_p of B_p.

Take a point $(\rho, \underline{\delta})$ on $Y(U^p, \overline{\rho})$. The problem of companion forms seeks to determine the set of characters $\underline{\delta}'$ of T_p, denoted by $W(\rho)$, such that pairs $(\rho, \underline{\delta}')$ appear on $Y(U^p, \overline{\rho})$. The existence of such *companion points* $(\rho, \underline{\delta}')$ is closely related to the appearance of certain irreducible locally analytic representations of G_p explicitly determined by $\underline{\delta}'$ and ρ, which we call *companion constituents*, inside $\Pi(\rho)^{\mathrm{an}}$. The existence of such companion constituents is a special case of the locally analytic socle conjecture of Breuil [17, 16]. For $v \in S_p$, we let $\rho_v := \rho|_{\mathrm{Gal}(\overline{F_v^+}/F_v^+)}$. The general recipe for $W(\rho)$ has been conjectured by Hansen [47] which depends only on those local Galois representations ρ_v for $v \in S_p$ and the notion of *trianguline representations* introduced by Colmez [26]. One could view the problem of companion forms or locally analytic socles as a locally analytic analogue of the weight part of Serre's modularity conjecture.

Let $D_{\mathrm{rig}}(\rho_v)$ be the étale (φ,Γ)-module over the Robba ring associated with ρ_v for $v\in S_p$. In the p-adic local Langlands program, locally analytic representations of p-adic Lie groups are expected to be related to (φ,Γ)-modules over the Robba rings which is the case for $\mathrm{GL}_2(\mathbb{Q}_p)$ by Colmez [27, V]. Beyond the foundational works of Kisin, Colmez and Emerton for $\mathrm{GL}_2(\mathbb{Q}_p)$ [58, 26, 40], we know in general and especially in our setting by the global triangulation results of Liu [62] or Kedlaya-Porttharst-Xiao [56] that the non-triviality of the Borel Emerton's Jacquet module $J_{B_p}(\Pi(\rho)^{\mathrm{an}})$ (i.e., in the finite slope case) implies that ρ_v is trianguline, i.e., $D_{\mathrm{rig}}(\rho_v)$ admits a full filtration

$$\mathrm{Fil}^{\bullet}D_{\mathrm{rig}}(\rho_v):D_{\mathrm{rig}}(\rho_v)=\mathrm{Fil}^{n}D_{\mathrm{rig}}(\rho_v)\supsetneq\cdots\supsetneq\mathrm{Fil}^{1}D_{\mathrm{rig}}(\rho_v)\supsetneq\mathrm{Fil}^{0}D_{\mathrm{rig}}(\rho_v)=\{0\}\tag{1.1}$$

of sub-(φ,Γ)-modules such that the graded pieces are rank one (φ,Γ)-modules.

Under the Taylor-Wiles hypothesis on $\overline{\rho}$, Breuil-Hellmann-Schraen proved in [20] the existence of all companion forms for regular generic crystalline points. In this paper, we generalize their results to non-regular generic crystalline points. To be precise, we take a point $(\rho,\underline{\delta})\in Y(U^p,\overline{\rho})$. We say ρ (or the point $(\rho,\underline{\delta})$) is crystalline if for all $v\in S_p$, ρ_v is crystalline. If ρ is crystalline, let $(\varphi_{v,i})_{i=1,\dots,n}$ be the eigenvalues of φ^{f_v} where φ is the crystalline Frobenius acting on $D_{\mathrm{cris}}(\rho_v)$ and $q_v=p^{f_v}$ is the cardinality of the residue field of F_v^+. Then we say ρ (or the point $(\rho,\underline{\delta})$) is *generic* if for any $v\in S_p$, $\varphi_{v,i}\varphi_{v,j}^{-1}\notin\{1,q_v\}$ for $i\neq j$. Assume that ρ is generic crystalline. A refinement $\mathcal{R}_v$ of ρ_v is a choice of an ordering of the pairwise distinct eigenvalues $\varphi_{v,1},\dots,\varphi_{v,n}$ and a refinement $\mathcal{R}=(\mathcal{R}_v)_{v\in S_p}$ of ρ is a choice of a refinement $\mathcal{R}_v$ for each $v\in S_p$. In fact the refinements $\mathcal{R}_v$ correspond to triangulations of $D_{\mathrm{rig}}(\rho_v)$ as (1.1) by [6]. Then the conjectural set of characters $W(\rho)$ admits a partition $W(\rho)=\coprod_{\mathcal{R}}W_{\mathcal{R}}(\rho)$ where $W_{\mathcal{R}}(\rho)=\prod_{v\in S_p}W_{\mathcal{R}_v}(\rho_v)$ and each $W_{\mathcal{R}_v}(\rho_v)$ is a finite set which can be explicitly described by $\mathcal{R}_v$ and ρ. Remark that the partition of $W(\rho)$ according to the refinements is also the partition under the equivalence relation that $\underline{\delta}\sim\underline{\delta}'$ if and only if $\underline{\delta}^{-1}\underline{\delta}'$ is a $\mathbb{Q}_p$-algebraic character of T_p. Our main theorem is the following.

THEOREM 1.1 (Theorem 4.12). – *Assume that U^p is small enough and assume the Taylor-Wiles hypothesis (cf. §4.2): F is unramified over F^+, F doesn't contain non-trivial p-th root of unity, U^p is hyperspecial at any finite place of F^+ that is inert in F and* $\overline{\rho}(\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1})))$ *is adequate. Let $(\rho,\underline{\delta})\in Y(U^p,\overline{\rho})$ be a point such that ρ is generic crystalline. Then there exists a refinement $\mathcal{R}$ of ρ such that $\underline{\delta}\in W_{\mathcal{R}}(\rho)$ and for any $\underline{\delta}'\in W_{\mathcal{R}}(\rho)$, the point $(\rho,\underline{\delta}')$ exists on $Y(U^p,\overline{\rho})$. Moreover, all the companion constituents associated with $W_{\mathcal{R}}(\rho)$ appear in $\Pi(\rho)$.*

REMARK 1.2. – In [20], the above theorem was proved under the extra assumption that for each $v\in S_p$, the Hodge-Tate weights of ρ_v are regular (pairwise distinct). But a stronger version was proved in [20]: in regular cases, $(\rho,\underline{\delta}')$ exists on $Y(U^p,\overline{\rho})$ for any refinement $\mathcal{R}'$ of ρ and $\underline{\delta}'\in W_{\mathcal{R}'}(\rho)$. This stronger result is easy to get from Theorem 1.1 in regular cases using locally algebraic vectors in $\Pi(\rho)$ and is not available in this paper for general crystalline points due to the non-existence of non-zero locally algebraic vectors in $\Pi(\rho)$ when ρ is non-regular (the non-existence can be seen using the results on infinitesimal characters in [34]). See Remark 5.24 for a partial result. The existence of all companion points in generic non-regular crystalline cases will need other methods.

The method in [20] was firstly replacing the eigenvariety $Y(U^p,\overline{\rho})$ by a larger *patched eigenvariety* $X_p(\overline{\rho})$ in [19, 18] constructed from the patching module in [22]. The patching