

quatrième série - tome 57 fascicule 5 septembre-octobre 2024

*ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Responsable du comité de rédaction / Editor-in-chief

YVES DE CORNULIER

Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE
de 1883 à 1888 par H. DEBRAY
de 1889 à 1900 par C. HERMITE
de 1901 à 1917 par G. DARBOUX
de 1918 à 1941 par É. PICARD
de 1942 à 1967 par P. MONTEL

Comité de rédaction au 7 juin 2024

S. CANTAT D. HÄFNER
G. CARRON D. HARARI
Y. CORNULIER Y. HARPAZ
F. DÉGLISE C. IMBERT
B. FAYAD A. KEATING
J. FRESÁN P. SHAN
G. GIACOMIN

Rédaction / Editor

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annaes@ens.fr

Édition et abonnements / Publication and subscriptions

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € (\$ 1 048). Vente au numéro : 77 €.

© 2024 Société Mathématique de France, Paris

En application de la loi du 1^{er} juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).

All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 n^{os} / an

ISING MODEL AND S-EMBEDDINGS OF PLANAR GRAPHS

BY DMITRY CHELKAK

ABSTRACT. – We discuss the notion of s-embeddings $\mathcal{S} = \mathcal{S}_{\mathcal{X}}$ of planar graphs carrying a nearest-neighbor Ising model. The construction of $\mathcal{S}_{\mathcal{X}}$ is based upon a choice of a global complex-valued solution \mathcal{X} of the propagation equation for Kadanoff-Ceva fermions. Each choice of \mathcal{X} provides an interpretation of all other fermionic observables as s-holomorphic functions on $\mathcal{S}_{\mathcal{X}}$. We set up a general framework for the analysis of such functions on s-embeddings \mathcal{S}^{δ} with $\delta \rightarrow 0$. Throughout this analysis, a key role is played by the functions \mathcal{Q}^{δ} associated with \mathcal{S}^{δ} , the so-called origami maps in the bipartite dimer model terminology. In particular, we give an interpretation of the mean curvature of the limit of discrete surfaces $(\mathcal{S}^{\delta}; \mathcal{Q}^{\delta})$ viewed in the Minkowski space $\mathbb{R}^{2,1}$ as the mass in the Dirac equation describing the continuous limit of the model.

We then focus on the simplest situation when \mathcal{S}^{δ} have uniformly bounded lengths/angles and $\mathcal{Q}^{\delta} = O(\delta)$; as a particular case this includes all critical Ising models on doubly periodic graphs via their canonical s-embeddings. In this setup we prove RSW-type crossing estimates for the random cluster representation of the model and the convergence of basic fermionic observables. The proof relies upon a new strategy and also provides a quantitative estimate on the speed of convergence.

RÉSUMÉ. – Nous introduisons la notion de s-plongements $\mathcal{S} = \mathcal{S}_{\mathcal{X}}$ de graphes planaires munis d'un modèle d'Ising. La construction de $\mathcal{S}_{\mathcal{X}}$ se base sur un choix d'une solution \mathcal{X} à valeurs complexes de l'équation de propagation pour les fermions de Kadanoff-Ceva. Chaque choix de \mathcal{X} donne une interprétation de toutes les autres observables fermioniques comme fonctions s-holomorphes sur $\mathcal{S}_{\mathcal{X}}$. Nous établissons un cadre général pour l'analyse de telles fonctions sur les s-plongements \mathcal{S}^{δ} avec $\delta \rightarrow 0$. Tout au long de cette analyse, un rôle essentiel est joué par les fonctions \mathcal{Q}^{δ} associées aux \mathcal{S}^{δ} , dites applications origami dans la terminologie du modèle de dimères sur des graphes bipartis. En particulier nous donnons une interprétation de la courbure moyenne de la limite des surfaces discrètes $(\mathcal{S}^{\delta}; \mathcal{Q}^{\delta})$ vue dans l'espace de Minkowski $\mathbb{R}^{2,1}$ comme la masse dans l'équation de Dirac décrivant la limite continue du modèle.

Par la suite nous nous concentrons sur la situation la plus simple où les \mathcal{S}^{δ} ont des longueurs/angles bornés uniformément et $\mathcal{Q}^{\delta} = O(\delta)$; cela comprend comme cas particulier tous les modèles d'Ising critiques sur des graphes doublement périodiques via leurs s-plongements canoniques. Dans ce cadre, nous démontrons des estimations de croisement de type RSW pour la représentation de Fortuin-Kasteleyn du modèle et la convergence des observables fermioniques les plus simples. La démonstration se base sur une nouvelle stratégie et fournit une estimation quantitative de la vitesse de convergence.

1. Introduction, main results and perspectives

1.1. General context

The Ising model of a ferromagnet, introduced by Lenz in 1920, recently celebrated its centenary; probably, this is one of the most studied models in statistical mechanics. The *planar* Ising model (i.e., the 2D model with nearest-neighbor interactions) gives rise to surprisingly rich structures of correlation functions; we refer the reader to monographs [26, 44, 46, 48], lecture notes [24], as well as to the introductions of the papers [1, 18] and references therein for more background on the subject, discussed from a variety of perspectives.

In this paper we consider the planar Ising model without the magnetic field, which is known to be exactly solvable on any graph and at any temperature: the partition function can be written as the Pfaffian of a certain matrix and the entries of the inverse matrix—known under the name *fermionic observables*—satisfy a simple *propagation equation*; we refer the reader to the paper [9] for more details on various combinatorial formalisms used to study the planar Ising model during its long history.

We prefer to work with the ferromagnetic Ising model defined on *faces* of a planar graph G ; the partition function is given by

$$(1.1) \quad \mathcal{Z}(G) = \sum_{\sigma: G^\circ \rightarrow \{\pm 1\}} \exp \left[\beta \sum_{e \in E(G)} J_e \sigma_{v_-^\circ(e)} \sigma_{v_+^\circ(e)} \right],$$

where G° and $E(G)$ denote the dual graph and the set of edges of G , respectively; $\beta = 1/kT$ is the inverse temperature, $J_e > 0$ are interaction constants assigned to the edges of G , and $v_\pm^\circ(e) \in G^\circ$ denote the two faces adjacent to $e \in E(G)$. Passing to the *domain walls* representation (also known as the low-temperature expansion) of the model, one can rewrite $\mathcal{Z}(G)$ as

$$\mathcal{Z}(G) = 2 \prod_{e \in E(G)} (x(e))^{-1/2} \times \sum_{C \in \mathcal{E}(G)} \prod_{e \in C} x(e), \quad x(e) := \exp[-2\beta J_e].$$

where $\mathcal{E}(G)$ denotes the set of all even subgraphs of G . Throughout this paper we identify edges of G with faces $z(e)$ of the graph $\Lambda(G) := G \cup G^\circ$ and denote

$$(1.2) \quad \theta_{z(e)} := 2 \arctan x(e) \in (0, \frac{1}{2}\pi),$$

Let us emphasize that we do *not* fix an embedding of G into \mathbb{C} at this point, thus θ_e is nothing more than another (abstract, i.e., not geometric) *parametrization* of the interaction constants J_e .

We are mostly interested in the situation when (G, x) is a big weighted graph carrying critical or near-critical Ising weights $x(e)$ though we do *not* precise the exact sense of this condition and mention it here only to give a proper perspective. For instance, the reader can think about the setup in which G is a subgraph of an infinite doubly periodic graph, in this case the criticality condition on the collection of weights $x(e)$ is known explicitly [41, 19]. However, this periodic setup is *not* the main motivation for our paper and should be viewed as a very particular case though our results are new even in this situation and, in particular, answer a question posed by Duminil-Copin and Smirnov in their lecture notes on the conformal invariance of 2D lattice models; see [24, Question 8.5].

Given a weighted graph (G, x) we aim to embed it into the complex plane \mathbb{C} (actually, we construct both an embedding of $\Lambda(G) := G \cup G^\circ$ and of its dual graph $\diamond(G)$) in a

way allowing to analyze (subsequential) limits of fermionic observables in the same spirit as in the seminal work of Smirnov [53, 54] on the critical Ising model on \mathbb{Z}^2 . A possible analogy could be Tutte's barycentric embeddings which, among other things, provide a framework to study the convergence of discrete harmonic functions to continuous ones. Let us emphasize that *s-embeddings* introduced in our paper are *not* directly related to these barycentric embeddings; in fact both can be viewed as special cases of a more general construction called *t-embeddings* or *Coloumb gauges* appearing in the bipartite dimer model context [39, 15]. The notion of s-embeddings was first announced in [8] and motivated a closely related research project [15, 16]. We benefit from this interplay and partly rely upon results obtained in [15] in a more general context.

One of the most important (from our perspective) motivations to study special embeddings of irregular weighted planar graphs into \mathbb{C} is the conjectural convergence of critical random maps carrying a lattice model to an object called *Liouville quantum gravity*; e.g., see [25, 27] or [51] and references therein. Though this discussion goes far beyond the scope of this paper, let us briefly mention several ideas of this kind that appeared in the literature: circle packing embeddings (see [47] and references therein), embeddings as piecewise flat Riemann surfaces (e.g., see [29] or [27, Section 1.7.1]), embeddings by square tilings for UST-decorated random maps, Tutte's embeddings [32, 31] and circle packings [30] for the mating-of-trees approach to the LQG, the so-called Cardy embeddings [33] introduced recently in the pure gravity context, etc. We believe that adding s-embeddings to this list might be a proper path for the analysis of random maps carrying the Ising model.

Besides generalizing the results of [54, 18] to a large family of deterministic graphs with 'flat' functions $\mathcal{Q}^\delta = O(\delta)$ and suggesting a general framework to attack problems coming from the random maps context, this paper also contains the following contributions to the subject:

- A new 'constructive' strategy of the proof of Theorem 1.2, which avoids compactness arguments in the Carathéodory topology on the set of planar domains and gives a quantitative estimate of the speed of convergence; see Section 4.3 for details. Due to a general framework recently developed in [52] this also allows to control the speed of convergence of FK-Ising interfaces to SLE(16/3) curves; see [52] for more details.
- In the non-flat case $\mathcal{Q}^\delta \not\rightarrow 0$ as $\delta \rightarrow 0$, we reveal the importance of embeddings of planar graphs carrying a nearest-neighbor Ising model into the *Minkowski space* $\mathbb{R}^{2,1}$ and not simply into the complex plane. As briefly discussed in Section 2.7, this leads to a rather unexpected interpretation of the *mass* appearing in the effective description of the model in the small mesh size limit as the *mean curvature* of the corresponding surface in $\mathbb{R}^{2,1}$.

The latter observation can be used in deterministic setups (e.g., see a recent paper [14] where such an interpretation of a near-critical model on isoradial grids is discussed) and leads to interesting questions on limits of random surfaces in $\mathbb{R}^{2,1}$ obtained via s-embeddings of appropriate random maps carrying the Ising model.