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SHARP ISOPERIMETRIC COMPARISON ON NON-COLLAPSED SPACES WITH LOWER RICCI BOUNDS

BY Gioacchino ANTONELLI, Enrico PASQUALETTO, Marco POZZETTA AND Daniele SEMOLA

ABSTRACT. – This paper studies sharp isoperimetric comparison theorems and sharp dimensional concavity properties of the isoperimetric profile for non-smooth spaces with lower Ricci curvature bounds, the so-called N-dimensional $\mathrm{RCD}(K, N)$ spaces.

The absence of most of the classical tools of geometric measure theory and the possible non-existence of isoperimetric regions on non-compact spaces are handled via an original argument to estimate first and second variation of the area for isoperimetric sets, avoiding any regularity theory, in combination with an asymptotic mass decomposition result of perimeter-minimizing sequences.

Most of our statements are new even for smooth, non-compact manifolds with lower Ricci curvature bounds and for Alexandrov spaces with lower sectional curvature bounds. They generalize several results known for compact manifolds, non-compact manifolds with uniformly bounded geometry at infinity, and Euclidean convex bodies.

RÉSUMÉ. – Cet article étudie les théorèmes de comparaison isopérimétrique et les propriétés de concavité du profil isopérimétrique pour les espaces non lisses à courbure de Ricci minorée: les espaces $\mathrm{RCD}(K, N)$ de dimension N.

L'absence de la plupart des outils classiques de théorie géométrique de la mesure et la non-existence possible de régions isopérimétriques dans les espaces non compacts sont traitées au moyen d'un argument original pour estimer la première et la deuxième variation de l'aire pour les ensembles isopérimétriques, en évitant la théorie de régularité. Cet argument est combiné avec un résultat de décomposition asymptotique de masse pour les suites minimisant le périmètre.

La plupart de nos énoncés sont nouveaux même pour les variétés lisses non compactes à courbure de Ricci minorée, et pour les espaces d'Alexandrov à courbure sectionnelle minorée. Ils généralisent plusieurs résultats connus pour les variétés compactes, les variétés non compactes à géométrie uniformément bornée à l'infini, et les corps convexes euclidiens.

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1. Introduction

Isoperimetry and lower Ricci curvature bounds

There is a celebrated connection between Ricci curvature and the isoperimetric problem in geometric analysis, going back at least to the Lévy-Gromov inequality [61, Appendix C]. The primary goal of this paper is to extend several results about the isoperimetric problem on compact Riemannian manifolds with lower Ricci curvature bounds to non-compact Riemannian manifolds and non-smooth spaces with lower Ricci curvature bounds. In order to deal with the possible non-existence of isoperimetric regions and with the lack of regularity we develop a series of new tools with respect to the classical literature. Non-smooth spaces enter into play naturally when dealing with smooth non-compact Riemannian manifolds, via the analysis of their pointed limits at infinity.

We consider the setting of N-dimensional $\mathrm{RCD}(K,N)$ metric measure spaces $(X,\mathsf{d},\mathscr{H}^N)$, for finite $N\in[1,\infty)$ and $K\in\mathbb{R}$, see [48, 68] after [89, 90, 73, 9, 54, 6, 50, 13, 39]. Here $K\in\mathbb{R}$ plays the role of (synthetic) lower bound on the Ricci curvature, $N\in[1,\infty)$ plays the role of (synthetic) upper bound on the dimension and $\mathscr{H}^N$ indicates the N-dimensional Hausdorff measure. This class includes (convex subsets of) smooth Riemannian manifolds with lower Ricci curvature bounds endowed with their volume measure, their noncollapsing measured Gromov-Hausdorff limits [45], and finite dimensional Alexandrov spaces with sectional curvature lower bounds [32, 86].

We shall rely on the theory of sets of finite perimeter in $\mathrm{RCD}(K,N)$ spaces, as developed in [3, 5, 30, 29]. For the sake of this introduction we just remark that it is fully consistent with the Euclidean and Riemannian ones. In particular, (reduced) boundaries of sets of finite perimeter are rectifiable, the perimeter coincides with the restriction of the $(N-1)$–dimensional Hausdorff measure to the (reduced) boundary and it does not charge the boundary of the ambient space.

Given an $\mathrm{RCD}(K,N)$ metric measure space $(X,\mathsf{d},\mathscr{H}^N)$ such that $\mathscr{H}^N(B_1(x))\geq v_0$ for any $x\in X$ for some $v_0>0$, we introduce the isoperimetric profile $I_X:[0,\mathscr{H}^N(X))\to[0,\infty)$ by

$$I_X(v):=\inf\left\{\mathrm{Per}(E)\,:\,E\subset X,\ \mathscr{H}^N(E)=v\right\},\tag{1.1}$$

where we drop the subscript X when there is no risk of confusion. When $E\subset X$ attains the infimum in (1.1) for $v=\mathscr{H}^N(E)$, we call it an isoperimetric region. In this setting we obtain:

- sharp second order differential inequalities for the isoperimetric profile, corresponding to equalities on the model spaces with constant sectional curvature. These inequalities are new even in the case of non-compact Riemannian manifolds and use in a crucial way the non-smooth approach. The proof bypasses the possible non-existence of isoperimetric regions on the space, that is classically used for such arguments, employing a concentration-compactness argument;
- a sharp Laplacian comparison theorem for the distance function from ∂E, which is a fundamental tool to prove the above items since it corresponds to the bounds usually obtained via first and second variation of the area in this low regularity setting;

— Gromov-Hausdorff stability and perimeters' convergence of isoperimetric regions along non-collapsing sequences of N-dimensional $\mathrm{RCD}(K,N)$ spaces. In order to prove these statements we deduce uniform regularity estimates for isoperimetric sets from uniform concavity estimates of the isoperimetric profiles.

Many of the above results are new even for smooth, non-compact manifolds with lower Ricci curvature bounds and for Alexandrov spaces with lower sectional curvature bounds. They answer several open questions in [23, 74, 71, 83, 21].

We expect the techniques developed in this paper to have a broad range of applications in geometric analysis under lower curvature bounds. For instance, in the study of more general geometric variational problems, the isoperimetric problem on weighted Riemannian manifolds with lower bounds on the Bakry-Émery curvature tensor, other geometric and functional inequalities.

Main results

On model spaces with constant sectional curvature $K/(N-1) \in \mathbb{R}$ and dimension $N \geq 2$ the isoperimetric profile $I_{K,N}$ solves the following second order differential equation on its domain:

$$-I''_{K,N} I_{K,N} = K + \frac{\left(I'_{K,N}\right)^2}{N-1}. \tag{1.2}$$

Equivalently, setting $\psi_{K,N} := I_{K,N}^{\frac{N}{N-1}}$, we have

$$-\psi''_{K,N} = \frac{KN}{N-1}\psi_{K,N}^{\frac{2-N}{N}}. \tag{1.3}$$

Combining the existence of isoperimetric regions for any volume, the regularity theory in geometric measure theory, and the second variation of the area (1.4), in [22, 23, 24, 80, 84] it was proved that the isoperimetric profile of a smooth, compact, N-dimensional Riemannian manifold with $\mathrm{Ric} \geq K$ verifies the inequality $\geq$ in (1.2) and (1.3) in a weak sense.

Here we obtain the following extension to the setting of $\mathrm{RCD}(K,N)$ metric measure spaces $(X, \mathsf{d}, \mathscr{H}^N)$ with a uniform lower bound on the volume of unit balls, without any assumption on the existence of isoperimetric regions. We stress again that the classical argument to show Theorem 1.1 in the compact setting uses in a crucial way the existence of isoperimetric regions for every volume, that we do not have at disposal in the present setting.

THEOREM 1.1 (cf. with Theorem 4.4). – *Let* $(X, \mathsf{d}, \mathscr{H}^N)$ *be an* $\mathrm{RCD}(K,N)$ *space. Assume that there exists* $v_0 > 0$ *such that* $\mathscr{H}^N(B_1(x)) \geq v_0$ *for every* $x \in X$.

Let $I : (0, \mathscr{H}^N(X)) \to (0, \infty)$ *be the isoperimetric profile of* X. *Then:*

1. *the inequality*

$$-I''I \geq K + \frac{(I')^2}{N-1} \quad \textit{holds in the viscosity sense on } (0, \mathscr{H}^N(X)),$$

2. *if* $\psi := I^{\frac{N}{N-1}}$ *then*

$$-\psi'' \geq \frac{KN}{N-1}\psi^{\frac{2-N}{N}} \quad \textit{holds in the viscosity sense on } (0, \mathscr{H}^N(X)).$$