ISSN 0012-9593

quatrième série - tome 58 fascicule 2 mars-avril 2025

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 494 euros.
Abonnement avec supplément papier :
Europe : 694 €. Hors Europe : 781 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 58, 2025, p. 275 à 388

HIGHER THETA SERIES FOR UNITARY GROUPS OVER FUNCTION FIELDS

BY TONY FENG, ZHIWEI YUN AND WEI ZHANG

ABSTRACT. – In previous work, we defined certain virtual fundamental classes for special cycles on the moduli stack of Hermitian shtukas, and related them to the higher derivatives of non-singular Fourier coefficients of Siegel-Eisenstein series. In the present article, we construct virtual fundamental classes in greater generality, including those expected to relate to the higher derivatives of *singular* Fourier coefficients. We assemble these classes into "higher" theta series, which we conjecture to be modular. Two types of evidence are presented: structural properties affirming that the cycle classes behave as conjectured under certain natural operations such as intersection products, and verification of modularity in several special situations. One innovation underlying these results is a new approach to special cycles in terms of derived algebraic geometry.

RÉSUMÉ. – Dans des travaux précédents, nous avons défini certaines classes fondamentales virtuelles pour des cycles spéciaux sur les champs de chtoucas hermitiens, et les avons liées aux dérivées supérieures des coefficients de Fourier non singuliers des séries de Siegel-Eisenstein. Dans cet article, nous construisons des classes fondamentales virtuelles dans des contextes plus généraux, y compris celles qui sont censées être liées aux dérivées supérieures des coefficients de Fourier singuliers. Nous assemblons ces classes en des séries thêta « supérieures », et conjecturons que ces séries thêtas sont modulaires. Deux types d'indications sont présentés en faveur de cette conjecture : des propriétés structurelles qui affirment que ces classes de cycles se comportent conformément à cette conjecture sous certaines opérations naturelles (par exemple des produits d'intersection), et la vérification de la modularité dans quelques situations spéciales. Ces résultats s'appuient sur une nouvelle approche des cycles spéciaux en termes de géométrie algébrique dérivée.

1. Introduction

The earliest examples of theta functions were generating series for the number of representations of integers by quadratic forms. It has been known at least since the work of Jacobi that theta functions enjoy remarkable symmetry properties, which later became known as *modularity*, that underlie many of their applications. An incarnation of theta functions in

doi:10.24033/asens.2606

arithmetic algebraic geometry was discovered by Kudla, who named them *arithmetic theta series*. This paper is about modularity in the context of arithmetic theta series.

The earliest examples of arithmetic theta series were constructed by Kudla as generating series with coefficients being cycle classes in the Chow groups of Shimura varieties [23]. Kudla envisioned a conjectural *arithmetic Siegel-Weil formula* [22], which would further require extending the special cycles to good integral models of Shimura varieties. A significant difficulty is the task of defining the appropriate cycle classes in the arithmetic Chow group indexed by singular Fourier coefficients. For example, for unitary Shimura varieties Kudla and Rapoport constructed the cycle classes on their integral models indexed by non-singular Fourier coefficients in [20, 21], while Li and the third author [27] proved an arithmetic Siegel-Weil formula for the non-singular Fourier coefficients (see also [28] for the orthogonal analog). However, the definition of the singular terms, and therefore also the full arithmetic theta series, remains open (except in some lower dimensional case, see [25]).

In [6] we proposed a function field analogue of this story: we defined special cycles on the moduli stack of Hermitian shtukas, constructed certain virtual fundamental classes for the cycles indexed by *non-singular* Fourier coefficients, and related them to the Taylor expansion of Fourier coefficients of corresponding Siegel-Eisenstein series. A novel feature of the function field version is that cycle classes can be defined for each non-negative integer r, and related to the r-th derivative of the Fourier coefficients of Siegel-Eisenstein series, whereas only the cases $r = 0$ and $r = 1$ seem to be witnessed over number fields (at least for the time being).

In this paper, we will construct virtual fundamental classes in general, going beyond the non-singular cases considered in [6], and assemble them into full "higher" arithmetic theta series (so named because they are related to higher derivatives of Siegel-Eisenstein series). The form of the singular terms exhibits interesting complexities that will be discussed further in §1.1. We formulate a conjecture about the modularity of such theta series, and then give evidence for this conjecture.

1.1. The modularity conjecture

We now introduce notation so as to be able to describe our conjecture and the main results with more precision. Let X be a smooth, proper and geometrically connected curve over $k = \mathbf{F}_q$ of characteristic $p \neq 2$, and let $\nu: X' \to X$ be a connected étale double cover, with the non-trivial automorphism denoted $\sigma \in \mathrm{Aut}(X'/X)$. Let F be the function field of X and let F' be the function field of X'. In [6] we defined the moduli stack $\mathrm{Sht}^r_{U(n)}$ parametrizing rank n "Hermitian shtukas" with r legs. We also defined certain special cycles $\mathcal{Z}^r_{\mathcal{E}}(a)$ indexed by $\mathcal{E}$, a vector bundle of rank m with $1 \leq m \leq n$ on X', and a Hermitian map $a: \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ where $\mathcal{E}^\vee := \mathscr{H}om(\mathcal{E}, \omega_{X'})$ is the Serre dual of $\mathcal{E}$. The space of such a was called $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$ in [6], but is called $\mathcal{A}_{\mathcal{E}}(k)$ in this paper. (Everything in [6] works in a slightly more general setup allowing a similitude factor, but for simplicity we omit this from our introduction.)

To define the higher theta series, we construct an appropriate virtual fundamental class $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}(a))$ for *every* $a \in \mathcal{A}_{\mathcal{E}}(k)$.

This was done in [6] when a is *non-singular* (meaning that $a: \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is injective as a map of coherent sheaves) and either $\mathrm{rank}\, \mathcal{E} = n$ or $\mathcal{E}$ is a direct sum of line bundles, by taking derived intersections from the situation where $\mathrm{rank}\, \mathcal{E} = 1$, following the ideas of [21] in the number field case. However, even in the non-singular case, to handle general m and $\mathcal{E}$ we must take a new approach based on Hitchin stacks (Definition 4.4). The dissimilarity to the number field situation comes from the fact that not every vector bundle on a proper curve splits as a sum of line bundles, while every vector bundle over the ring of integers of a number field splits as a direct sum of line bundles.

For singular a, the construction of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is more complicated. The cycle $\mathcal{Z}^r_{\mathcal{E}}(a)$ admits an open-closed decomposition according to the possible kernels of the map a, and the contribution from each stratum is the product of a virtual fundamental class constructed from a Hitchin stack and the top Chern class of a certain tautological bundle. The construction is completed in Definition 4.8. It may be a useful guide for the number field case, where no definition of special cycle classes in the arithmetic Chow group is currently known, for singular Fourier coefficients, at the time of this writing.

Having defined $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ for each a, we then assemble them into higher theta series. More precisely, if $\operatorname{rank} \mathcal{E} = m$, then we consider the quasi-split unitary group (with respect to the double cover X'/X) of rank $2m$ over X, abbreviated $U(2m)$, and the standard Siegel parabolic P_m. (In the main body of the paper, starting in §9.1, we use the notation H_m for $U(2m)$.) We write down a function on $U(2m)(\mathbb{A})$ valued in $\mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n)})$:

$$\widetilde{Z}^r_m : U(2m)(\mathbb{A}) \longrightarrow \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n)})$$

characterized by the following properties:

1. $\widetilde{Z}^r_m$ is left invariant under $P_m(F)$ and right invariant under $K = U(2m)(\widehat{\mathcal{O}})$;
2. for any point in $P_m(F)\backslash P_m(\mathbb{A})/K \cap P_m(\mathbb{A}) \simeq P_m(F)\backslash U(2m)(\mathbb{A})/K$ represented by $(\mathcal{G}, \mathcal{E})$, where $\mathcal{G}$ is a rank $2m$ vector bundle on X' with a skew-Hermitian structure $h : \mathcal{G} \simeq \sigma^*\mathcal{G}^*$ and $\mathcal{E}$ is a Lagrangian sub-bundle of $\mathcal{G}$, we have a "Fourier expansion" (in the sense of [6, §2.6])

$$\widetilde{Z}^r_m(\mathcal{G}, \mathcal{E}) = \chi(\det \mathcal{E}) q^{n(\deg \mathcal{E} - \deg \omega_X)/2} \sum_{a \in \mathcal{A}_{\mathcal{E}}(k)} \psi_0(\langle e_{\mathcal{G},\mathcal{E}}, a\rangle) \zeta_*[\mathcal{Z}^r_{\mathcal{E}}(a)]. \tag{1.1}$$

Here we refer to §4.6 for the undefined notation in the right hand side. We note that, in the special case $\mathcal{E} = \mathcal{O}_{X'}^{\oplus m}$ the trivial bundle of rank m, the set of all such $(\mathcal{G}, \mathcal{E})$ is naturally isomorphic to $N_m(F)\backslash N_m(\mathbb{A})/K \cap N_m(\mathbb{A})$, where N_m denotes the unipotent radical of P_m. Then $\mathcal{A}_{\mathcal{E}}(k)$ is naturally isomorphic to the Pontryagin dual of $N_m(F)\backslash N_m(\mathbb{A})/K \cap N_m(\mathbb{A})$ (depending on the choice of a non-trivial character $\psi_0 : k \to \mathbb{C}^\times$). For this $\mathcal{E}$, (1.1) more closely resembles the expressions for arithmetic theta series on Shimura varieties, as one finds for example in [23, (5.4)].

Conjecture 1.1 (Modularity conjecture). – *The function $\widetilde{Z}^r_m$ descends to a function*

$$Z^r_m : U(2m)(F)\backslash U(2m)(\mathbb{A}) \longrightarrow \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n)}),$$

i.e., $\widetilde{Z}^r_m$ is left $U(2m)(F)$-invariant.

In other words, the class $\widetilde{Z}^r_m(\mathcal{G}, \mathcal{E}) \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n)})$ should depend only on the Hermitian bundle $\mathcal{G}$ and not on its Lagrangian sub-bundle $\mathcal{E}$.