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THE MASSEY VANISHING CONJECTURE FOR FOURFOLD MASSEY PRODUCTS MODULO 2

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ABSTRACT. – We prove the Massey vanishing conjecture, due to Mináč and Tân, for $n = 4$ and $p = 2$. That is, we show that for all fields F, if a fourfold Massey product modulo 2 is defined over F, then it vanishes over F.

RÉSUMÉ. – On démontre la conjecture d'annulation de Massey, due à Mináč et Tân, pour $n = 4$ et $p = 2$. Autrement dit, on montre que pour tout corps F, si un produit quadruple de Massey modulo 2 est défini sur F, alors il s'annule sur F.

1. Introduction

Let (A, ∂) be a differential graded ring, that is, a cochain complex equipped with a graded associative product satisfying the Leibniz rule with respect to the differential ∂, and let $H^*(A)$ be the cohomology ring of A. For all integers $n \geq 2$ and all $a_1, \dots, a_n \in H^1(A)$, one may define the n-fold Massey product $\langle a_1, \dots, a_n \rangle$: it is a certain subset of $H^2(A)$. For $n = 2$, the Massey product $\langle a_1, a_2 \rangle$ is equal to the singleton $\{a_1 a_2\}$, but for $n \geq 3$ the Massey product $\langle a_1, \dots, a_n \rangle$ can be empty or contain more than one element. One says that $\langle a_1, \dots, a_n \rangle$ is defined if it is non-empty, and that it vanishes if it contains 0. (See the introduction of [8] for the precise definition of Massey product, which will not be needed in this paper.) We have the following implications:

$$\langle a_1, \dots, a_n \rangle \text{ vanishes} \Rightarrow \langle a_1, \dots, a_n \rangle \text{ is defined} \Rightarrow a_i a_{i+1} = 0 \quad (i = 1, \dots, n-1).$$

Massey [13] introduced Massey products in algebraic topology; in this case A is the singular cochain complex of a topological space. Massey proved that the Borromean rings are not equivalent to three unlinked circles by showing that the singular cochain complex of the complement of the Borromean rings in $\mathbb{R}^3$ admits a non-trivial triple Massey product.

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In this paper, we consider Massey products in Galois cohomology. Let p be a prime number, Γ be a profinite group and $A := C^*(\Gamma, \mathbb{Z}/p\mathbb{Z})$ be the differential graded $\mathbb{F}_p$-algebra of mod p continuous cochains of Γ. We write $H^*(\Gamma, \mathbb{Z}/p\mathbb{Z})$ for the cohomology algebra $H^*(A)$. When Γ is the absolute Galois group of a field F, we will write $H^*(F, \mathbb{Z}/p\mathbb{Z})$ for $H^*(\Gamma, \mathbb{Z}/p\mathbb{Z})$. Here and throughout the paper, we will view $\mathbb{Z}/p\mathbb{Z}$ as a trivial Γ-module, and in fact our main results concern coefficients $\mathbb{Z}/2\mathbb{Z}$.

Let $n \geq 2$ be an integer, and let $U_{n+1} \subset \mathrm{GL}_{n+1}(\mathbb{F}_p)$ be the p-Sylow subgroup of upper unitriangular matrices, that is, the upper triangular matrices all of whose diagonal entries are equal to 1. Let $Z_{n+1} \subset U_{n+1}$ be the subgroup generated by the matrix Z having 1 in each diagonal entry and in the entry $(1, n+1)$ and 0 elsewhere. Then $Z_{n+1} \simeq \mathbb{Z}/p\mathbb{Z}$ is the center of U_{n+1}. We let $\overline{U}_{n+1} := U_{n+1}/Z_{n+1}$; one may think of $\overline{U}_{n+1}$ as the group of upper unitriangular matrices with top-right corner removed. We obtain the following diagram of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\iota} & U_{n+1} & \longrightarrow & \overline{U}_{n+1} & \longrightarrow & 1 \\ & & & & & \overset{\varphi}{\searrow} & \downarrow{\scriptstyle\overline{\varphi}} & & \\ & & & & & & (\mathbb{Z}/p\mathbb{Z})^n, & & \end{array}$$

where the row is a central short exact sequence, $\iota(1 + p\mathbb{Z}) = Z$, the surjective homomorphism φ forgets the entries of all upper diagonals of an upper unitriangular matrix except for the first one, and $\overline{\varphi}$ is induced by φ.

Let $\chi_1, \dots, \chi_n \in H^1(\Gamma, \mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Z}/p\mathbb{Z})$, and write χ for the group homomorphism $(\chi_1, \dots, \chi_n)\colon \Gamma \to (\mathbb{Z}/p\mathbb{Z})^n$. Dwyer [1] proved that the Massey product $\langle \chi_1, \dots, \chi_n \rangle \subset H^2(\Gamma, \mathbb{Z}/p\mathbb{Z})$

– is defined if and only if χ lifts to $\overline{U}_{n+1}$, i.e., $\chi = \overline{\varphi} \circ \chi'$ for some homomorphism $\chi'\colon \Gamma \to \overline{U}_{n+1}$, and
– vanishes if and only if χ lifts to U_{n+1}, i.e., $\chi = \varphi \circ \chi''$ for some homomorphism $\chi''\colon \Gamma \to U_{n+1}$.

(The reader not familiar with the general definition of a Massey product may take the above as the definitions of the phrases "the Massey product is defined" and "the Massey product vanishes.")

In contrast with the situation in algebraic topology, Hopkins-Wickelgren [9] showed that, if F is a number field, all triple Massey products in $H^*(F, \mathbb{Z}/2\mathbb{Z})$ vanish as soon as they are defined. (In field theory, considerations related to triple Massey products had already appeared in an earlier article of Gao-Leep-Mináč-Smith [4, Theorems 3.7 and 4.7], although Massey products were not mentioned there.) The result of Hopkins-Wickelgren was extended to all fields F by Mináč-Tân [20]. It motivated the following conjecture, known as the Massey vanishing conjecture, which first appeared in [20] under an assumption on roots of unity, then in general in [18].

CONJECTURE 1.1 (Mináč-Tân). – *For every field F, every prime p, every integer $n \geq 3$ and all $\chi_1, \dots, \chi_n \in H^1(F, \mathbb{Z}/p\mathbb{Z})$, if the Massey product $\langle \chi_1, \dots, \chi_n \rangle \in H^2(F, \mathbb{Z}/p\mathbb{Z})$ is defined, then it vanishes.*

When p is invertible in F and F contains a primitive p-th root of unity, by Kummer theory the characters $\chi_1, \dots, \chi_n$ correspond to scalars $a_1, \dots, a_n \in F^\times$ uniquely determined up to p-th powers. One says that $\langle a_1, \dots, a_n \rangle$ is defined (resp. vanishes) when $\langle \chi_1, \dots, \chi_n \rangle$ is defined (resp. vanishes). Conjecture 1.1 then predicts that $\langle a_1, \dots, a_n \rangle$ vanishes as soon as it is defined.

Conjecture 1.1 is in the spirit of the *profinite inverse Galois problem*, i.e., of the fundamental question: Which profinite groups are realizable as absolute Galois groups of fields? Indeed, a historically fruitful approach to the profinite inverse Galois problem has been to give constraints on the cohomology of absolute Galois groups. The most spectacular example of this is the norm-residue isomorphism theorem (the Bloch-Kato conjecture), proved by Voevodsky and Rost. When F contains a primitive p-th root of unity, this theorem implies, in particular, that $H^*(F, \mathbb{Z}/p\mathbb{Z})$ is a quadratic algebra: it admits a presentation with generators in degree 1 and relations in degree 2. This property is false in general for arbitrary profinite groups, and so gives a way to prove that a profinite group does not arise as the absolute Galois group of a field.

From this point of view, Conjecture 1.1 predicts a new way in which the cohomology of absolute Galois groups is simpler than that of arbitrary profinite groups. Already the $n = 3$ case of Conjecture 1.1 yields remarkable restrictions on the profinite groups which can appear as absolute Galois groups; see for example the work of Efrat [2] and Mináč-Tân [19].

Since its formulation, Conjecture 1.1 has motivated a large body of work by many authors. It is known in a number of cases:

- when F is a number field, $n = 3$ and $p = 2$, by Hopkins-Wickelgren [9];
- when F is arbitrary, $n = 3$ and $p = 2$, by Mináč-Tân [20];
- when F is number field, $n = 3$ and p is odd, by Mináč-Tân [17];
- when F is arbitrary, $n = 3$ and p is odd, by Matzri [14], followed by Efrat-Matzri [3] and Mináč-Tân [18];
- when F is a number field, $n = 4$ and $p = 2$, by Guillot-Mináč-Topaz-Wittenberg [7];
- when F is a number field and n and p are arbitrary, by Harpaz-Wittenberg [8].

There are partial results for specific classes of fields; for example, rigid odd fields [16]. However, when F is an arbitrary field, very little is known beyond the $n = 3$ case. In this paper, we prove the case $n = 4$ and $p = 2$ of Conjecture 1.1, with no assumptions on F.

THEOREM 1.2. – *Conjecture 1.1 is true for $n = 4$ and $p = 2$. That is, for all fields F and all $\chi_1, \chi_2, \chi_3, \chi_4 \in H^1(F, \mathbb{Z}/2\mathbb{Z})$, if the mod 2 Massey product $\langle \chi_1, \chi_2, \chi_3, \chi_4 \rangle$ is defined, then it vanishes.*

The proof of Theorem 1.2 is different from those of Guillot-Mináč-Topaz-Wittenberg and of Harpaz-Wittenberg in the number field case, as the tools used by them (local-global principles, Brauer-Manin obstruction) are not available over an arbitrary field.

We sketch the proof of Theorem 1.2. If K is a field (or a product of fields) of characteristic different from 2, and $a, b \in K^\times$, we denote by $\mathrm{Br}(K)$ the Brauer group of K, and by $(a, b) \in \mathrm{Br}(K)$ the class of the quaternion algebra corresponding to a and b. We also set $K_a := K[x_a]/(x_a^2 - a)$ and $K_{a,b} := (K_a)_b$.