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TOPOLOGY OF IRRATIONALLY INDIFFERENT ATTRACTORS

BY Davoud CHERAGHI

Abstract. – We study the post-critical set of a class of holomorphic maps with an irrationally indifferent fixed point. We prove a trichotomy for the topology of the post-critical set based on the arithmetic of the rotation number at the fixed point. The only possibilities are Jordan curve, one-sided hairy Jordan curve, and Cantor bouquet. This explains the degeneration of the closed invariant curves inside the Siegel disks, as one varies the rotation number.

Résumé. – Nous étudions l'ensemble post-critique d'une classe d'applications holomorphes avec un point fixe indifférent irrationnel. Nous prouvons une trichotomie pour la topologie de l'ensemble post-critique basée sur l'arithmétique du nombre de rotation au point fixe. Les seules options sont une courbe de Jordan, une courbe de Jordan velue unilatérale et un bouquet de Cantor. Cela explique la dégénérescence des courbes invariantes fermées à l'intérieur des disques de Siegel, lorsque l'on fait varier le nombre de rotation.

1. Introduction

1.1. Irrationally indifferent attractors

Let

$$f(z) = e^{2\pi i\alpha} z + O(z^2) \tag{1.1}$$

be a germ of a holomorphic map defined near $0 \in \mathbb{C}$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The fixed point at 0 is called *irrationally indifferent*. It is known that the local dynamics of f near 0 depends on the arithmetic nature of α in a delicate fashion. By classical results of Siegel [67] and Brjuno [12], if α satisfies an arithmetic condition, now called *Brjuno type*, f is conformally conjugate to the rotation by $2\pi\alpha$ near 0. The maximal domain of linearization (conjugacy) is called the *Siegel disk* of f at 0, and is denoted by $\Delta(f)$ here. Within $\Delta(f)$, the local dynamics is trivial; any orbit in $\Delta(f)$ is dense in an invariant analytic Jordan curve. Yet, in a remarkable development [74], Yoccoz showed that if α is not a Brjuno number, the quadratic polynomial

$$P_\alpha(z) = e^{2\pi i\alpha} z + z^2$$

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is not linearizable at 0. Despite that, Perez-Marco [57] showed that there remains a non-trivial local invariant set at 0. However, the topology of the local invariant set, and the local dynamics near 0 remained mysterious, even for P_α. In this paper, for the first time, we explain the delicate topological structure of the (local) attractor, and the dynamics of the map on it.

When f is a polynomial or a rational function, the irrationally indifferent fixed point at 0 influences the global dynamics of f. By the classical results [30, 47], there is at least a recurrent critical point of f which "interacts" with the fixed point at 0. For any such critical point c_f, we define

$$\Lambda(c_f) = \overline{\bigcup_{i\geq 1} f^{\circ i}(c_f)}.$$

When f is linearizable at 0, the boundary of $\Delta(f)$ is contained in $\Lambda(c_f)$, and when f is not linearizable at 0, then $0 \in \Lambda(c_f)$. The set $\Lambda(c_f)$ is part of the post-critical set of f, which is the closure of the orbits of all critical values of f. By a general result in holomorphic dynamics [46], unless the Julia set is equal to the whole Riemann sphere, for Lebesgue almost every z in the Julia set of f, the spherical distance between $f^{\circ k}(z)$ and the post-critical set of f tends to 0 as $k \to \infty$.

For "badly approximable" α, $\Lambda(c_f)$ is well understood over the last four decades. The main method is an ingenious surgery procedure, which is introduced by Douady [28] for quadratic polynomials, Zakeri [77] for cubic polynomials, Shishikura (unpublished work) for all polynomials, and Zhang [78] for all rational functions. Through the surgery, the problem is linked to the dynamics of analytic circle maps, where the works of Herman, Yoccoz and Swianek [38, 72, 69] play a key role. The culmination of those works shows that when α is bounded type and f is a rational function, $c_f \in \partial\Delta(f)$ and $\Lambda(c_f) = \partial\Delta(f)$ is a quasi-circle (a Jordan curve with controlled geometry). In [51], McMullen developed a renormalization method to show that, among other features, when α is an algebraic number, $\Lambda(c_f)$ enjoys rescaling self-similarity at c_p. In a far reaching generalization in the quadratic case, Petersen and Zakeri [60] employed trans quasi-conformal surgery to show that for almost every α, $c_{P_\alpha} \in \partial\Delta(P_\alpha)$ and $\Lambda(c_{P_\alpha}) = \partial\Delta(P_\alpha)$ is a David circle (a generalization of quasi-circle).

For "well-approximable" α, the structure of $\Lambda(c_f)$ remained mostly mysterious, despite few sporadic surprising results such as [58, 3]. Computer simulations suggest that large entries in the continued fraction of α result in oscillations of the invariant curves in $\Delta(f)$. The size of an entry and its location in the continued fraction of α, as well as non-linearity of large iterates of f, result in an intricate oscillation of the invariant curves in $\Delta(f)$. See Figure 1. For α with infinitely many extremely large entries, the consecutive oscillations may degenerate the closed invariant curves. For a class of maps, we explain the degeneration of invariant curves under perturbations of α.

1.2. Statements of the results

Inou and Shishikura in [40] introduced a sophisticated renormalization scheme $(\mathcal{F}, \mathcal{R})$, where $\mathcal{F}$ is an infinite dimensional class of maps as in (1.1), and $\mathcal{R} : \mathcal{F} \to \mathcal{F}$ is a renormalization operator. Every $f \in \mathcal{F}$ has a certain covering structure, and a (preferred) critical point c_f. The set $\mathcal{F}$ contains (the restriction to a neighborhood of 0 of) some polynomials

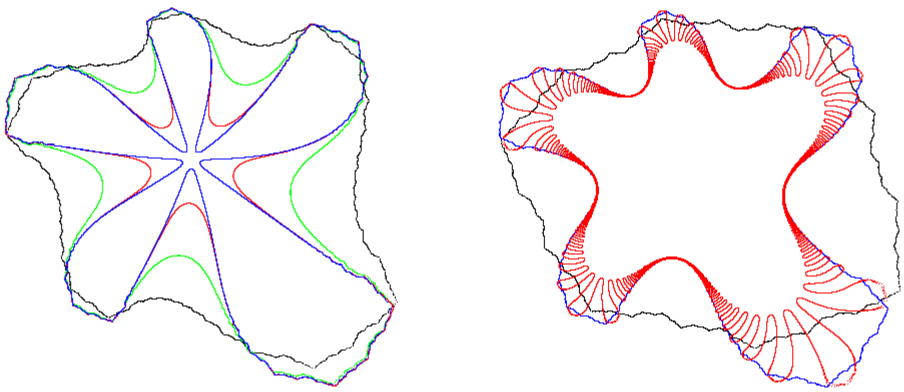

FIGURE 1. Left image: computer simulations of the orbit of c_{P_α} for rotation numbers $\alpha = [2, 2, \overline{2}]$, $[2, 2, 10^2, \overline{2}]$, $[2, 2, 10^4, \overline{2}]$, and $[2, 2, 10^8, \overline{2}]$. Right image: computer simulations of the orbit of c_{P_α} for $\alpha = [2, 2, \overline{2}]$, $[2, 2, 10^2, \overline{2}]$, and $[2, 2, 10^2, 10^8, \overline{2}]$.

and rational functions of arbitrarily large degrees. The scheme requires α to be of sufficiently *high type*, that is, α belongs to the set

$$\mathrm{HT}_N = \{\varepsilon_0/(a_0 + \varepsilon_1/(a_1 + \varepsilon_2/(a_2 + \cdots))) \mid \forall n \geq 0, a_n \geq N, \varepsilon_n = \pm 1\},$$

for a suitable N. In HT_N, there are α of bounded type, as well as α with arbitrarily large entries.

The scheme $(\mathcal{F}, \mathcal{R})$ was successfully employed by Inou and Shishikura to trap the orbit of c_f in a dynamically defined neighborhood of 0. Moreover, they showed that the orbit of c_f is infinite, there are no periodic points in $\Lambda(c_f)$, and in particular, $\Lambda(c_{P_\alpha})$ is not equal to the Julia set of P_α.

In [18, 19] we carried out a detailed quantitative analysis of the renormalization scheme $(\mathcal{F}, \mathcal{R})$, and obtained fine estimates on the changes of coordinates which appear in the renormalization. In [20], we built a toy model for the renormalization of maps with an irrationally indifferent fixed point. We employ those methods to explain the delicate structure of $\Lambda(c_f)$.

THEOREM A (Trichotomy of irrationally indifferent attractors). – *There is $N \geq 2$ such that for every $\alpha \in \mathrm{HT}_N$ and every $f(z) = e^{2\pi i \alpha} z + O(z^2)$ in the Inou-Shishikura class $\mathcal{F}$, one of the following holds:*

(i) *α is Herman type, and $\Lambda(c_f)$ is a Jordan curve enclosing* 0*,*

(ii) *α is Brjuno but not Herman type, and $\Lambda(c_f)$ is a one-sided hairy Jordan curve enclosing* 0*,*

(iii) *α is not Brjuno type, and $\Lambda(c_f)$ is a Cantor bouquet at* 0*.*

The trichotomy also holds for the quadratic polynomials P_α, when $\alpha \in \mathrm{HT}_N$.

The set of *Herman numbers* was discovered by Herman and Yoccoz [38, 73] in their landmark studies of the dynamics of analytic circle diffeomorphisms. In this paper we do not make any connections to circle maps—the Brjuno and Herman types naturally come up.