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NON-SYMMETRIC QUANTUM LOOP GROUPS AND K-THEORY

BY MICHELA VARAGNOLO AND ÉRIC VASSEROT

ABSTRACT. – We realize the quantum loop groups and shifted quantum loop groups of arbitrary types, possibly non-symmetric, using critical K-theory. This generalizes the Nakajima construction of symmetric quantum loop groups via quiver varieties to non-symmetric types. We also give a new geometric construction of some simple modules of both quantum loop groups and shifted quantum loop groups.

RÉSUMÉ. – Nous réalisons les groupes quantiques affines et les groupes quantiques affines décalés de type arbitraire, éventuellement non symétrique, en utilisant la K-théorie critique. Cette construction généralise la construction de Nakajima des groupes quantiques affines symétriques via les variétés de carquois. Nous donnons également une nouvelle construction géométrique de certains modules simples des groupes quantiques affines et des groupes quantiques affines décalés.

1. Introduction and notation

1.1. Introduction

The quiver varieties $\mathfrak{M}(W)$, the graded quiver varieties $\mathfrak{M}^\bullet(W)$, and their Steinberg varieties $\mathcal{Z}(W)$ and $\mathcal{Z}^\bullet(W)$ were introduced by Nakajima in [30, 31] and [32]. The K-theory of the Steinberg varieties, equipped with a convolution product, yields a family of algebras $K(\mathcal{Z}^\bullet(W))$ which are closely related to the quantum loop groups $\mathrm{U}_\zeta(L\mathfrak{g})$ of symmetric types. This algebra is important for the finite dimensional modules of $\mathrm{U}_\zeta(L\mathfrak{g})$ and their q-characters, see [33, 34]. Nakajima's geometric realization of $\mathrm{U}_\zeta(L\mathfrak{g})$ in $K(\mathcal{Z}^\bullet(W))$ does not extend to quantum groups of non-symmetric (finite or Kac-Moody) types. How to generalize [32] is an old question. An alternative approach has been suggested recently in [35] using Coulomb branches. The Coulomb realization also yields some interpretation of finite dimensional simple modules, but it is of a different nature from the construction here. In this paper we introduce a new family of convolution algebras attached to quiver varieties with potentials, called critical convolution algebras. Here the K-theory is replaced

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by the critical K-theory, which is the Grothendieck group of the derived factorization categories attached to LG-models considered in [3, 9, 21, 20]. The critical K-theory depends on the choice of some function (the trace of the potential). It is supported on the critical set of this function. We prove that the critical convolution algebra attached to the Steinberg variety $\widetilde{\mathcal{Z}}^\bullet(W)$ of the graded triple quiver variety $\widetilde{\mathfrak{M}}^\bullet(W)$ yields a geometric realization of all non-twisted quantum loop groups and of all non-twisted negatively shifted quantum loop groups for some well chosen potentials. More precisely we prove the following theorem.

THEOREM 1.1. – *Let $\mathfrak{g}$ be any simple complex Lie algebra.*

(a) *There is an algebra homomorphism* $\mathrm{U}_\zeta(L\mathfrak{g}) \to K\big(\widetilde{\mathfrak{M}}^\bullet(W)^2, (\tilde{f}^\bullet_\gamma)^{(2)}\big)_{\widetilde{\mathcal{Z}}^\bullet(W)}$ *and representations of* $\mathrm{U}_\zeta(L\mathfrak{g})$ *on* $K(\widetilde{\mathfrak{M}}^\bullet(W), \tilde{f}^\bullet_\gamma)_{\widetilde{\mathfrak{L}}^\bullet(W)}$ *and* $K(\widetilde{\mathfrak{M}}^\bullet(W), \tilde{f}^\bullet_\gamma)$.

(b) *There is an algebra homomorphism* $\mathrm{U}^{-w}_\zeta(L\mathfrak{g}) \to K\big(\widehat{\mathfrak{M}}^\bullet(W)^2, (\widehat{f}^\bullet_2)^{(2)}\big)_{\widehat{\mathcal{Z}}^\bullet(W)}$ *and representations of* $\mathrm{U}^{-w}_\zeta(L\mathfrak{g})$ *on* $K(\widehat{\mathfrak{M}}^\bullet(W), \widehat{f}^\bullet_2)_{\widehat{\mathfrak{L}}^\bullet(W)}$ *and* $K(\widehat{\mathfrak{M}}^\bullet(W), \widehat{f}^\bullet_2)$.

Here $\tilde{f}^\bullet_\gamma$, $\widehat{f}^\bullet_2$ are potentials on $\widetilde{\mathfrak{M}}^\bullet(W)$, $\widehat{\mathfrak{M}}^\bullet(W)$, and $\widehat{\mathfrak{M}}^\bullet(W)$ is a simply framed version of $\widetilde{\mathfrak{M}}^\bullet(W)$. The algebra $\mathrm{U}^{-w}_\zeta(L\mathfrak{g})$ is the shifted quantum loop group. Note that the 0-shifted quantum loop group is $\mathrm{U}_\zeta(L\mathfrak{g})$ up to some central elements. The potential $\tilde{f}^\bullet_\gamma$ is a deformation of a potential $\tilde{f}^\bullet_1$ on $\widetilde{\mathfrak{M}}^\bullet(W)$ depending on a deformation parameter γ. Part (a) of the theorem can be viewed as an extension of the construction in [32] because:

- for symmetric type, there is an isomorphism of $\mathfrak{M}^\bullet(W)$ with the critical set $\mathrm{crit}(\tilde{f}^\bullet_1)$ of a function $\tilde{f}^\bullet_1 : \widetilde{\mathfrak{M}}^\bullet(W) \to \mathbb{C}$ which will be defined later in the text,
- in Proposition 4.6, a dimension reduction yields an algebra and a module isomorphisms

$$K\big(\widetilde{\mathfrak{M}}^\bullet(W)^2, (\tilde{f}^\bullet_1)^{(2)}\big)_{\widetilde{\mathcal{Z}}^\bullet(W)} = K\big(\mathcal{Z}^\bullet(W)\big), \quad K\big(\widetilde{\mathfrak{M}}^\bullet(W), \tilde{f}^\bullet_1\big) = K\big(\mathfrak{M}^\bullet(W)\big).$$

For symmetric types, the function $\tilde{f}^\bullet_1$ comes from some cubic potential, while non-symmetric types require potentials of higher degrees. The possibility to use potentials of arbitrary degree is an important property of critical convolution algebras which has no analogue for the Nakajima convolution algebras. The potential we use for non-symmetric types appears already in the work of Hernandez-Leclerc and Geiss-Leclerc-Schröer [19, 14] on cluster algebras, or in the work of Yang-Zhao [46] on cohomological Hall algebras. We expect the general theory of Nakajima in [32] to generalize to all types using critical cohomology and K-theory. We will come back to this elsewhere. In particular in the non-shifted case, we have the following conjecture, see the text below for the notation.

CONJECTURE 1.2. – *For each* $w \in \mathbb{N}I^\bullet$ *the* $\mathrm{U}_\zeta(\mathrm{L}\mathfrak{g})$*-modules*

$$K\big(\widetilde{\mathfrak{M}}^\bullet(W), \tilde{f}^\bullet_1\big), \quad K\big(\widetilde{\mathfrak{M}}^\bullet(W), \tilde{f}^\bullet_1\big)_{\widetilde{\mathfrak{L}}^\bullet(W)}$$

are isomorphic to the costandard module and the standard module with ℓ-highest weight Ψ_w.

For fundamental modules, this conjecture follows from Theorem 1.3 below.

Another important point is that the Nakajima construction permits to recover the classification of the simple finite dimensional modules of quantum loop groups, but it does not give a geometric construction of those. More precisely, the K-theory of quiver varieties yields a geometric realization of the standard modules, and the simple modules are the

Jordan Hölder constituents of the standards. Remarkably, varying the potentials, the critical K-theory also gives a realization of the simple modules in several settings: we realize both all Kirillov-Reshetikhin modules of usual quantum loop groups and the prefundamental modules of shifted quantum loop groups as the critical K-theory of some LG-models attached to quivers. This construction is new already for symmetric types. It was partly motivated by the work of Liu in [27], where some representations of some shifted quantum loop groups are realized via the K-theory of quasi-maps spaces. Liu's construction uses some limit procedure similar to the limit procedure of Hernandez-Jimbo in [18]. In our setting given by critical K-theory of triple quiver varieties, this limit procedure admits a natural interpretation. More precisely, we prove the following.

THEOREM 1.3. – (a) *The Kirillov-Reshetikhin modules of the quantum loop group are realized in the critical K-theory of graded triple quiver varieties (for a convenient choice of the parameter γ).*

(b) *The negative prefundamental modules of the shifted quantum loop group are realized in the critical K-theory of graded triple quiver varieties.*

A further motivation comes from cluster theory. Using cluster algebras, Hernandez-Leclerc give in [19] a q-character formula for prefundamental and Kirillov-Reshetikhin representations in terms of Euler characteristic of quiver Grassmanians. Their character formula does not give any geometric realization of the (shifted) quantum loop group action. It is surprising that our construction yields indeed a representation of the (shifted) quantum loop group in some critical K-theory groups supported on the same quiver Grassmanians. The Kirillov-Reshetikhin modules are particular cases of reachable modules for the cluster algebra structure on the Grothendieck ring of the quantum loop group considered in [24]. The Euler characteristic description of the q-characters extends to all reachable modules. We expect that all reachable modules admit also a realization in critical K-theory.

Finally let us point out a link with K-theoretic Hall algebras of a quiver with potential. These algebras were introduced by Padurariu in [38]. It was proved there that Isik's Koszul duality (= dimensional reduction) implies that the K-theoretic Hall algebras of triple quivers with some particular potential coincide with the K-theoretic Hall algebras of preprojective algebras considered in [44]. We define an algebra homomorphism from K-theoretic Hall algebras to K-theoretic critical convolution algebras using Hecke correspondences. As a consequence, the K-theoretic critical convolution algebras may be viewed as some doubles of the K-theoretic Hall algebras. These doubles are a better setting for representation theory than the K-theoretic Hall algebras, as the examples below suggest. Different doubles of the same K-theoretic Hall algebras can be realized via different K-theoretic critical convolution algebras. This is especially transparent in the symmetric case, since:

– in [44] we proved that twisted K-theoretic Hall algebras of preprojective Dynkin quivers are isomorphic to the Drinfeld halves of quantum loop groups,

– quantum loop groups and shifted quantum loop groups map to (different) K-theoretic critical convolution algebras by Theorems 4.1 and 4.7.