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THE ÉTALE LOCAL STRUCTURE OF ALGEBRAIC STACKS

BY Jarod ALPER, Jack HALL and David RYDH

Abstract. – We prove that an algebraic stack with affine stabilizers over an arbitrary base is étale-locally a quotient stack around any point with a linearly reductive stabilizer. This generalizes earlier work by the authors (stacks over algebraically closed fields) and by Abramovich, Olsson and Vistoli (stacks with finite inertia). In addition, we prove a number of foundational results, which are new even over a field. These include various coherent completeness and effectivity results for adic sequences of algebraic stacks. Finally, we give several applications of our results and methods, such as structure theorems for linearly reductive group schemes and generalizations to the relative setting of Sumihiro's theorem on torus actions and Luna's étale slice theorem.

Résumé. – Soit X un champ algébrique, localement de présentation finie et quasi-séparé sur un espace algébrique quasi-séparé, avec stabilisateurs affines. Nous montrons que tout point de X, avec stabilisateur linéairement réductif, possède un voisinage étale qui est un champ quotient. Ce résultat généralise les travaux précédents des auteurs (champs sur un corps algébriquement clos) et d'Abramovich, Olsson et Vistoli (champs avec inertie finie). En outre, nous obtenons plusieurs résultats fondamentaux qui sont nouveaux même sur un corps. Ceux-ci comprennent divers résultats de complétion cohérente d'un champ algébrique le long d'un sous-champ fermé, ainsi que d'effectivité de déformations formelles d'un champ algébrique. Enfin, nous fournissons plusieurs applications de ces résultats, notamment des généralisations au cadre relatif du théorème de Sumihiro sur les actions de tores et du théorème du « slice étale » de Luna.

1. Introduction

This paper offers a broad generalization and extension of our previous work [9], which provided a local structure theorem for algebraic stacks of finite type over an algebraically closed field. In addition to establishing a local structure theorem for algebraic stacks defined over an arbitrary base (Theorem 1.1), we prove a number of foundational results that are new even over an algebraically closed field. This includes a general coherent completeness result for algebraic stacks (Theorem 1.6), which becomes particularly powerful when coupled with

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Tannaka duality (see §1.7). We also prove an effectivity theorem for adic sequences of noetherian algebraic stacks (Theorem 1.10), analogous to Grothendieck's result on algebraization of formal schemes [27, III.5.4.5]. While of independent interest, this is one of the key ingredients for the other main theorems in this paper, including the local structure theorem.

We prove several other foundational results and provide numerous applications to equivariant geometry and moduli theory. To highlight a few, we prove that adequate moduli spaces are universal for maps to algebraic spaces (Theorem 3.12); this implies that GIT quotients in positive characteristic are categorical quotients in algebraic spaces, which was formerly unknown, even over an algebraically closed field. We establish that various properties (e.g., the resolution property[(1)]) and objects (e.g., morphisms, finite étale covers, vector bundles), defined on a closed substack $\mathcal{X}_0$ of $\mathcal{X}$, extend to an étale neighborhood of $\mathcal{X}_0$ if there is an affine good moduli space $\mathcal{X} \to X$ (see Theorems 7.17 to 7.20). These extension results are technical but extremely useful in local-to-global arguments. We also prove a generalization of Sumihiro's theorem on torus actions (Theorem 10.2): an algebraic space X over a base S with an action of a torus $T \to S$ has T-equivariant affine étale neighborhoods.

1.1. A local structure theorem

We prove that if $\mathcal{X}$ is an algebraic stack satisfying some mild assumptions, then a point $x \in |\mathcal{X}|$ with linearly reductive stabilizer has an étale neighborhood $([\operatorname{Spec} A/\mathrm{GL}_n], w) \to (\mathcal{X}, x)$ inducing an isomorphism on residual gerbes. This is the conclusion of the following theorem in the special case where $\mathcal{W}_0 = \mathcal{G}_x$.

THEOREM 1.1 (Local structure). – *Let S be a quasi-separated algebraic space and $\mathcal{X}$ an algebraic stack, locally of finite presentation, and quasi-separated over S, with affine stabilizers. Let $x \in |\mathcal{X}|$ be a point with residual gerbe $\mathcal{G}_x$ and image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite. Let $h_0\colon \mathcal{W}_0 \to \mathcal{G}_x$ be a smooth (resp. étale) morphism where $\mathcal{W}_0$ is a gerbe over the spectrum of a field and has linearly reductive stabilizer. Then there exist an algebraic stack $\mathcal{W} = [\operatorname{Spec} A/\mathrm{GL}_n]$ and a point $w \in |\mathcal{W}|$ with an identification $\mathcal{G}_w = \mathcal{W}_0$ together with a cartesian diagram*

$$\begin{array}{ccc} \mathcal{G}_w = \mathcal{W}_0 & \xrightarrow{h_0} & \mathcal{G}_x \\ \downarrow & & \downarrow \\ [\operatorname{Spec} A/\mathrm{GL}_n] = \mathcal{W} & \xrightarrow{h} & \mathcal{X}, \end{array}$$

where $h\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ is a smooth (resp. étale) pointed morphism and w is closed in its fiber over s. Moreover, if $\mathcal{X}$ has separated (resp. affine) diagonal and h_0 is representable, then h can be arranged to be representable (resp. affine).

The flexibility of allowing $\mathcal{W}_0 \to \mathcal{G}_x$ to be a smooth or étale morphism (rather than an isomorphism) is useful when the stabilizer G_x of x is not linearly reductive, which is a particularly restrictive condition in positive characteristic. Assuming that the residual gerbe $\mathcal{G}_x = BG_x$ is trivial, one can choose any linearly reductive subgroup $H \subseteq G_x$

(1) That is, every quasi-coherent sheaf is a quotient of a direct sum of vector bundles. If the algebraic stack is quasi-compact and quasi-separated with affine stabilizers, then this is equivalent to being expressible as $[U/\mathrm{GL}_n]$, where U is a quasi-affine scheme [59, 26].

such that G_x/H is smooth (resp. étale) and apply the theorem to the smooth (resp. étale) morphism $\mathcal{W}_0 = BH \to BG_x$.

REMARK 1.2 (Known results). – In the case where $\mathcal{X}$ has finite inertia and h_0 is an isomorphism, this theorem had been established in [1, Thm. 3.2]. When $S = \operatorname{Spec} k$ and k is algebraically closed, this theorem follows from [9, Thm. 1.1], which established the stronger result that $\mathcal{W}$ can be written as $[\operatorname{Spec} A/H]$.

REMARK 1.3 (Further generalizations). – In Theorem 8.1 and Theorem 8.2, we provide a more refined description of the stack $\mathcal{W}$ as a quotient stack $[\operatorname{Spec} A/G]$ for a specific group scheme G in terms of properties of the gerbe $\mathcal{W}_0$. For instance, if the stabilizer group of $\mathcal{W}_0$ is connected, we can arrange that G is split reductive, and if $\mathcal{W}_0 = BG_0$ is neutral, we can arrange that G is a deformation of G_0 which is linearly reductive under mild characteristic assumptions.

In work with Halpern-Leistner [7], this theorem is generalized to allow the case where $\mathcal{W}_0$ is an arbitrary linearly fundamental stack (rather than a gerbe over a field), where $x \in |\mathcal{X}|$ is an arbitrary point (rather than the finiteness of $\kappa(x)/\kappa(s)$), and where $\mathcal{W}_0 \to \mathcal{G}_x$ is syntomic (rather than smooth or étale).

The proof of Theorem 1.1 is given in Section 5 and follows the same general strategy as the proof of [9, Thm. 1.1]:

1. We begin by constructing smooth infinitesimal deformations $h_n \colon \mathcal{W}_n \to \mathcal{X}_n$ where $\mathcal{X}_n$ is the n-th infinitesimal neighborhood of $\mathcal{G}_x$ in $\mathcal{X}$.
2. We show that the system $\mathcal{W}_n$ effectivizes to a *coherently complete* stack $\widehat{\mathcal{W}}$. This is Theorem 1.10.
3. Tannaka duality [33] (see also §1.7) then gives us a unique formally smooth morphism $\widehat{h} \colon \widehat{\mathcal{W}} \to \mathcal{X}$.
4. Finally we apply equivariant Artin algebraization [9, App. A] to approximate $\widehat{h}$ with a smooth morphism $h \colon \mathcal{W} \to \mathcal{X}$.

Step (1) follows by standard infinitesimal deformation theory. Step (2) is the main technical result of this paper. Theorem 1.10 is far more general than the related results in [9], even over an algebraically closed field. Step (3) was handled in [33]. Steps (1)–(3) are summarized in Theorem 1.11. The equivariant Artin algebraization results established in [9, App. A] are sufficient for step (4).

REMARK 1.4 (Existence theorem). – Theorem 1.1 and its refinements are fundamental ingredients in the recent article of the first author with Halpern-Leistner and Heinloth on establishing necessary and sufficient conditions for an algebraic stack to admit a good moduli space [10].