ISSN 0012-9593

quatrième série - tome 59 fascicule 1 janvier-février 2026

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Ian GLEASON

On the geometric connected components of moduli spaces of p-adic shtukas and local Shimura varieties

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 514 euros.
Abonnement avec supplément papier :
Europe : 722 €. Hors Europe : 813 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 59, 2026, p. 199 à 272

ON THE GEOMETRIC CONNECTED COMPONENTS OF MODULI SPACES OF p-ADIC SHTUKAS AND LOCAL SHIMURA VARIETIES

BY IAN GLEASON

ABSTRACT. – We study topological properties of moduli spaces of p-adic shtukas and local Shimura varieties. On one hand, we construct and study the specialization map for moduli spaces of p-adic shtukas at parahoric level whose target is an affine Deligne-Lusztig variety. On the other hand, given a p-adic shtuka datum (G, b, μ), with G unramified over $\mathbb{Q}_p$ and such that (b, μ) is HN-irreducible, we determine the set of geometric connected components of infinite level moduli spaces of p-adic shtukas. In other words, we understand $\pi_0(\mathrm{Sht}_{G,b,\mu,\infty} \times \mathrm{Spd}\,\mathbb{C}_p)$ with its right $G(\mathbb{Q}_p) \times G_b(\mathbb{Q}_p) \times W_E$-action. As a corollary, we prove new cases of a conjecture of Rapoport and Viehmann.

RÉSUMÉ. – Nous étudions les propriétés topologiques des espaces de modules de chtoucas p-adiques et des variétés de Shimura locales. D'une part, nous construisons et étudions l'application de spécialisation pour les espaces de modules de chtoucas p-adiques au niveau parahorique dont la cible du morphisme est une variété affine de Deligne-Lusztig. D'autre part, étant donné une donnée de chtouca p-adique (G, b, μ), avec G non ramifié sur $\mathbb{Q}_p$ et tel que (b, μ) soit HN-irréductible, nous déterminons l'ensemble des composantes connexes géométriques des espaces de modules de niveau infini de chtoucas p-adiques. En d'autres termes, nous comprenons $\pi_0(\mathrm{Sht}_{G,b,\mu,\infty} \times \mathrm{Spd}\,\mathbb{C}_p)$ avec son action de $G(\mathbb{Q}_p) \times G_b(\mathbb{Q}_p) \times W_E$ à droite. En corollaire, nous prouvons de nouveaux cas d'une conjecture de Rapoport et Viehmann.

Introduction

In [52], Rapoport and Viehmann propose that there should be a theory of p-adic local Shimura varieties. They conjecture the existence of towers of rigid-analytic spaces whose cohomology groups "understand" the local Langlands correspondence for general p-adic reductive groups. In this way, these towers of rigid-analytic varieties would "interact" with the local Langlands correspondence in a manner similar to how Shimura varieties "interact" with the global Langlands correspondence. Moreover, they conjecture many properties and compatibilities that these towers should satisfy (see [52, § 5]).

 doi:10.24033/asens.2638

In the last decade, the theory of local Shimura varieties has gone through a drastic transformation with Scholze's introduction of perfectoid spaces and the theory of diamonds [55]. In [57], Scholze and Weinstein construct the sought-after towers of rigid-analytic spaces and generalize them to what are now known as moduli spaces of p-adic shtukas (or mixed characteristic local shtukas) [57, §23.1]. Moreover, since then, many of the expected properties and compatibilities for local Shimura varieties have been verified and generalized to moduli spaces of p-adic shtukas. The study of the geometry and cohomology of local Shimura varieties and moduli spaces of p-adic shtukas is still a very active area of research due to its connection to the local Langlands correspondence. One of the main aims of this article is to study the set of geometric connected components of moduli spaces of p-adic shtukas (see Definition 3.3)

$$\pi_0(\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,\infty})$$

attached to a p-adic shtuka datum (G, b, μ) (as in Section 3.1.4), and to describe the right action of the group $G(\mathbb{Q}_p) \times G_b(\mathbb{Q}_p) \times W_E$ on $\pi_0(\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,\infty})$. This set controls the first cohomology group of moduli spaces of p-adic shtukas. The upshot is that connected components of moduli spaces of p-adic shtukas are completely described by local class field theory (see Theorem 3.19 for a precise statement). As a consequence of our results, we settle [52, Conjecture 4.30] in the case of unramified groups (see Theorem 3.19).

Let us recall the formalism of local Shimura varieties and moduli spaces of p-adic shtukas. Let $\mathbb{C}_p \supseteq \mathbb{Q}_p$ denote a completed algebraic closure of $\mathbb{Q}_p$ and let $\breve{\mathbb{Q}}_p \subseteq \mathbb{C}_p$ denote the completion of the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. A local p-adic shtuka datum over $\mathbb{Q}_p$ is a triple (G, b, μ) where G is a reductive group over $\mathbb{Q}_p$, μ is a conjugacy class of geometric cocharacters $\mu : \mathbb{G}_m \to G_{\bar{\mathbb{Q}}_p}$, and b is an element of Kottwitz' set $B(G, \mu)$ [40, § 6]. Whenever μ is minuscule, we say that (G, b, μ) is a local Shimura datum (see [52, Definition 5.1]). We let $E/\mathbb{Q}_p$ denote the reflex field of μ and we let $\breve{E} = E \cdot \breve{\mathbb{Q}}_p$ be the compositum inside $\mathbb{C}_p$. Associated to (G, b, μ), there is a tower of diamonds over $\mathrm{Spd}\, \breve{E}$, denoted $(\mathrm{Sht}_{G,b,\mu,K})_K$, where $K \subseteq G(\mathbb{Q}_p)$ ranges over compact open subgroups of $G(\mathbb{Q}_p)$ [57, § 23.3]. Moreover, whenever μ is minuscule, $\mathrm{Sht}_{G,b,\mu,K}$ is represented by the diamond associated to a unique smooth rigid-analytic space $\mathbb{M}_K$ over $\breve{E}$. The tower $(\mathbb{M}_K)_K$ is the local Shimura variety [57, Definition 24.1.3].

Associated to $b \in B(G, \mu)$ there is a reductive group G_b over $\mathbb{Q}_p$ [(1)] §3.1.3. After base change to a completed algebraic closure, each individual space

$$\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,K} := \mathrm{Sht}_{G,b,\mu,K} \times \mathrm{Spd}\, \mathbb{C}_p$$

comes equipped with continuous and commuting right actions by $G_b(\mathbb{Q}_p)$ and the Weil group W_E. Moreover, the tower receives a right action by the group $G(\mathbb{Q}_p)$ by using correspondences. When we take the limit as $K \subseteq G(\mathbb{Q}_p)$ shrinks we obtain the space at infinite level, which we denote $\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,\infty}$. Overall, this space comes equipped with a right action by the group $G(\mathbb{Q}_p) \times G_b(\mathbb{Q}_p) \times W_E$ §3.1.5.

The group $G(\mathbb{Q}_p) \times G_b(\mathbb{Q}_p) \times W_E$ acts continuously on $\pi_0(\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,\infty})$ and one of our main theorems (see Theorem 3.19) describes explicitly this action under two technical

(1) In the literature, the group that we denote G_b is often denoted by J_b.

assumptions on the triple (G, b, μ). The first assumption is that G is an unramified reductive group i.e., G is a quasi-split connected reductive group over $\mathbb{Q}_p$ whose base change to some unramified extension $\mathbb{Q}_{p^s}$ becomes split (e.g., all split groups are unramified). The second assumption is that the pair (b, μ) is HN-irreducible (Hodge-Newton irreducible). For $G = \mathrm{GL}_n$ this condition asks in rough terms that the Hodge polygon determined by μ and the Newton polygon determined by b do not meet at a "breaking point" (see Definition 3.7 for the precise definition). For unramified groups, our result is optimal in a sense which we discuss later in this introduction (see also Remark 3.21). It is also likely that the condition that G is unramified can be removed (see Remark 3.22).

Before stating the main theorem of §3 we set some notation. Let (G, b, μ) be local p-adic shtuka datum with G an unramified reductive group over $\mathbb{Q}_p$. Let G^{der} denote the derived subgroup of G, G^{sc} denote the simply connected cover of G^{der}, consider the image of $G^{\mathrm{sc}}(\mathbb{Q}_p)$ in $G(\mathbb{Q}_p)$ and let $G(\mathbb{Q}_p)_\circ = G(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$. The group $G(\mathbb{Q}_p)_\circ$ is a locally profinite topological group and it is the maximal abelian quotient of $G(\mathbb{Q}_p)$ when this latter is considered as an abstract group (see Remark 3.18). Let $\mathrm{Art}_E : W_E \to E^\times$ be the reciprocity character from local class field theory. In §3.3.2 we associate to μ and to b continuous maps of topological groups $\mathrm{Nm}_\mu^{E,\circ} : E^\times \to G(\mathbb{Q}_p)_\circ$ and $\det_b^\circ : G_b(\mathbb{Q}_p) \to G(\mathbb{Q}_p)_\circ$ respectively.

THEOREM 1 (Theorem 3.19). – *Let (G, b, μ) be a p-adic shtuka datum such that G is an unramified reductive group over $\mathbb{Q}_p$ and such that the pair (b, μ) is HN-irreducible. Let E denote the reflex field of μ. Then the following hold.*

1. *The right $G(\mathbb{Q}_p)$-action on $\pi_0(\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,\infty})$ is trivial on $\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$ and the corresponding G°-action is simply-transitive.*
2. *If $s \in \pi_0(\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,\infty})$ and $j \in G_b(\mathbb{Q}_p)$ then*
$$s \star_{G_b} j = s \star_{G(\mathbb{Q}_p)_\circ} \det_b^\circ(j^{-1}).$$
3. *If $s \in \pi_0(\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,\infty})$ and $\gamma \in W_E$ then*
$$s \star_{W_E} \gamma = s \star_{G(\mathbb{Q}_p)_\circ} [\mathrm{Nm}_\mu^{E,\circ} \circ \mathrm{Art}_E(\gamma)].$$

REMARK 1. – In the particular case in which $G^{\mathrm{der}} = G^{\mathrm{sc}}$ the above theorem is established by constructing an equivariant isomorphism
$$\pi_0(\mathrm{Sht}^{\mathrm{geo}}_{G,b,\mu,\infty}) \simeq \pi_0(\mathrm{Sht}^{\mathrm{geo}}_{G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty}),$$
where $(G^{\mathrm{ab}}, b^{\mathrm{ab}}, \mu^{\mathrm{ab}})$ is the p-adic shtuka datum attached to the maximal abelian quotient of G. This case settles [52, Conjecture 4.30] of Rapoport and Viehmann in the case in which the group is unramified (see Corollary 3.13).

REMARK 2. – Since moduli spaces of p-adic shtukas (as with most moduli spaces) do not have an explicit presentation, saying concrete things about their geometry is usually hard. This is to be expected since the reason we study moduli spaces of p-adic shtukas is to get a better understanding of the local Langlands correspondence, which is itself a very deep statement. Although we do not discuss explicit applications of our theorem in this article, we believe that our result is not only hard but also powerful. To convince the reader about this, we recall that the study of connected components of affine Deligne-Lusztig varieties [11] was one of the key ingredient in Kisin's work on integral models of Shimura varieties [33].