ISSN 0012-9593

quatrième série - tome 59 fascicule 2 mars-avril 2026

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Danny OFEK

Lower bounds on the essential dimension of reductive groups

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 514 euros.
Abonnement avec supplément papier :
Europe : 722 €. Hors Europe : 813 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 59, 2026, p. 331 à 385

LOWER BOUNDS ON THE ESSENTIAL DIMENSION OF REDUCTIVE GROUPS

BY DANNY OFEK

ABSTRACT. – We introduce a technique for proving lower bounds on the essential dimension of split reductive groups. As an application, we strengthen the best previously known lower bounds for various split simple algebraic groups, most notably for the exceptional group E_8. In the case of the projective linear group PGL_n, we recover A. Merkurjev's celebrated lower bound with a simplified proof. Our technique relies on decompositions of loop torsors over valued fields due to P. Gille and A. Pianzola.

RÉSUMÉ. – Nous introduisons une méthode pour établir des bornes inférieures sur la dimension essentielle d'un groupe réductif déployé. À titre d'application, nous renforçons les meilleures bornes inférieures connues jusqu'à présent pour divers groupes algébriques simples déployés, notamment pour le groupe exceptionnel E_8. Dans le cas du groupe projectif linéaire PGL_n, nous redémontrons, par une preuve simplifiée, la célèbre borne inférieure due à A. Merkurjev. Notre approche repose sur les décompositions de torseurs de lacets sur des corps valués, dues à P. Gille et A. Pianzola.

1. Introduction

Let G be a smooth linear algebraic group over a field k_0 and $\gamma \in H^1(L, G)$ a G-torsor over a field $k_0 \subset L$. A *field of definition* for γ is a subfield $k_0 \subset F \subset L$ such that γ lies in the image of the natural map:

$$H^1(F, G) \to H^1(L, G).$$

The *essential dimension* of γ is the minimal number of parameters needed to define γ. It is given by the formula:

$$\mathrm{ed}(\gamma) = \min\left\{ \mathrm{trdeg}_{k_0}(F) \mid F \text{ is a field of definition of } \gamma \right\}.$$

The *essential dimension* of G is defined as the supremum $\mathrm{ed}(G) = \sup\{\mathrm{ed}(\gamma)\}$ taken over all overfields $k_0 \subset L$ and torsors $\gamma \in H^1(L, G)$. It should be thought of as the minimal

Danny Ofek was partially supported by a graduate fellowship from the University of British Columbia.

 doi:10.24033/asens.2640

number of algebraically independent parameters needed to define an arbitrary G-torsor. Often G-torsors correspond bijectively to a class of algebraic objects, in which case $\mathrm{ed}(G)$ is the minimal number of independent parameters needed to define a member of that class. For example:

— The essential dimension of the symmetric group S_d is the number of independent variables required to define an arbitrary separable field extension of degree d. Equivalently, it is the number of parameters needed to define a generic polynomial of degree d up to Tschirnhaus transformations. In this form, mathematicians have tried to compute it as early as the 17th century. For an overview, see [24, 10, 9].

— The essential dimension of PGL_d is the number of independent variables required to define an arbitrary central division algebra of degree d. This quantity has been studied since generic division algebras were first defined by C. Procesi [52, Section 2].

The problem of computing $\mathrm{ed}(G)$ for a general algebraic group G has been studied by many authors since it was first posed by J. Buhler-Z. Reichstein [9] and Reichstein [53]. For a comprehensive survey of the developments in the field, we refer the reader to [45].

Let p be a prime integer. The *essential dimension at p* of G, denoted $\mathrm{ed}(G; p)$, measures how many parameters are required to construct an arbitrary G-torsor up to prime-to-p extensions. See Section 2 for a precise definition. The inequality $\mathrm{ed}(G) \geq \mathrm{ed}(G; p)$ always holds, and almost all existing techniques to prove lower bounds on $\mathrm{ed}(G)$ apply to $\mathrm{ed}(G; p)$ as well; see [54, Section 5]. The same is true of the technique introduced in this paper.

1.1. Overview of previous techniques

Z. Reichstein-B. Youssin gave the first systematic way to prove lower bounds on the essential dimension of algebraic groups. We recall their main theorem, commonly referred to as "the fixed-point method" because it was proven by an analysis of fixed points on generically free G-varieties. See [28, 14] for generalizations to positive characteristic.

THEOREM 1.1 ([56, Theorem 7.7]). – *Assume G is defined over an algebraically closed field of characteristic zero k_0 and G° is semisimple. Let $A \subset G(k_0)$ be a finite abelian group and p a prime number.*

1. *If $C_G(A \cap G^\circ)$ is finite, then* $\mathrm{ed}(G) \geq \mathrm{rank}(A)$.
2. *If A is a p-group and $C_G(A \cap G^\circ)$ is finite, then* $\mathrm{ed}(G; p) \geq \mathrm{rank}(A)$.

While Theorem 1.1 applies to many groups, it usually gives bounds which are far from tight. This is partially explained by the fact that the G-torsors witnessing the lower bound can be constructed over an iterated Laurent series field $k_0(\!(t_1)\!) \ldots (\!(t_r)\!)$ which is a relatively simple field when k_0 is algebraically closed (here $r = \mathrm{rank}(A)$).

P. Brosnan-Reichstein-A. Vistoli [7, 8] and later N. Karpenko-A. Merkurjev [33] introduced stack-theoretic techniques to construct G-torsors of high essential dimension over function fields of Severi-Brauer varieties. Stack-theoretic techniques give much stronger lower bounds than is possible using Theorem 1.1 for some groups, like finite p-groups and algebraic tori [33, 37]. However, they give trivial lower bounds for most semisimple groups, including all adjoint simple groups. In [44], Merkurjev overcame this limitation of the stack-theoretic methods for adjoint groups of type A_n by proving $\mathrm{ed}(\mathrm{PGL}_n; p) \geq \mathrm{ed}(T; p)$ for a certain torus T. He then computed $\mathrm{ed}(T; p)$ using [37] to obtain the lower bounds

$$\mathrm{ed}(\mathrm{PGL}_{p^r}; p) \geq (r-1)p^r + 1, \tag{1.1}$$

which are orders of magnitude stronger than the lower bounds previously obtained by Theorem 1.1. Merkurjev's arguments are specific to groups of type A_n because they rely on explicit computations in the Brauer group (see [13] for generalizations to SL_n/μ_d). We note that a proof of (1.1) for $p = r = 2$ was first announced by M. Rost [57]. Merkurjev proved (1.1) for $r = 2$ and p arbitrary by different means in [43].

1.2. Main results

In this paper, we introduce a technique to prove lower bounds on $\mathrm{ed}(G), \mathrm{ed}(G; p)$, which applies whenever G° is split reductive. We proceed in two steps:

1. We first prove $\mathrm{ed}(G) \geq \mathrm{ed}(C_G(A))$ for any finite diagonalizable subgroup $A \subset G$ satisfying certain conditions.
2. We choose A in a systematic way, so that $C_G(A)$ is an extension of a torus by a finite group. This allows us to apply the results of [38] to give a strong lower bound on $\mathrm{ed}(C_G(A))$.

In this way we obtain new lower bounds on the essential dimension of some simple groups as well as recover (1.1), see Theorem 1.5 below and Section 14. The next theorem gives sufficient conditions for the inequality $\mathrm{ed}(G) \geq \mathrm{ed}(C_G(A))$ to hold. Note that we do not assume G° is split. Thus the first step in our technique works more generally and the assumption that G° is split is only needed for the second step. Recall that a G-torsor $[c_\sigma] \in H^1(F, G)$ is called *anisotropic* if the twisted group ${}_cG$ contains no copy of $\mathbb{G}_m$. A finite algebraic group A over k_0 is called *diagonalizable*, if it is isomorphic to $\mu_{n_1} \times \cdots \times \mu_{n_r}$ for some $n_1, \ldots, n_r$ coprime to $\mathrm{char}\, k_0$.

THEOREM 1.2. – *Let G be a smooth linear algebraic group over a field k_0. Assume either k_0 is perfect or G° is reductive. Let $A \subset G$ be a finite diagonalizable subgroup.*

1. *Let $p \neq \mathrm{char}\, k_0$ be a prime. If A is a p-group and $C_G(A)$ admits an anisotropic torsor over some p-closed field $k_0 \subset k$, then we have:*
$$\mathrm{ed}(G; p) \geq \mathrm{ed}(C_G(A); p).$$
2. *Assume* $\mathrm{char}\, k_0$ *is good for G (see Definition 2.1). If $C_G(A)$ admits an anisotropic torsor over some field $k_0 \subset k$, then we have:*
$$\mathrm{ed}(G) \geq \mathrm{ed}(C_G(A)).$$

Let F be a Henselian valued field with value group of finite rank. Our proof of Theorem 1.2 relies on the decompositions of *loop torsors* over F. Loop torsors and their decompositions were introduced by P. Gille-A. Pianzola for iterated Laurent series over a characteristic zero field in the context of the classification of loop algebras [27]. We will use both [27] and the recent generalizations to valuation rings of positive characteristic obtained by Gille [26].