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ALGEBRAIZATION TECHNIQUES AND RIGID-ANALYTIC ARTIN-GROTHENDIECK VANISHING

BY OFER GABBER AND BOGDAN ZAVYALOV

ABSTRACT. – We prove an algebraization result for rig-smooth algebras over a general noetherian ring. The result says that we can always algebraize a geometrically reduced affinoid rigid-analytic space in "one direction" in an appropriate sense. As an application of this result, we show the remaining cases of the Artin-Grothendieck vanishing for affinoid algebras. This allows us to deduce a stronger version of the rigid-analytic Artin-Grothendieck vanishing conjecture over a field of characteristic 0. Using a completely different set of ideas, we also obtain a weaker version of this conjecture over a field of characteristic $p > 0$.

RÉSUMÉ. – Nous prouvons un résultat d'algébrisation pour les algèbres rig-smooth sur un anneau noethérien général. Ce résultat affirme que nous pouvons toujours algébriser un espace analytique rigide affinoïde géométriquement réduit dans «une direction» au sens approprié. En application de ce résultat, nous montrons les cas restants du théorème d'annulation d'Artin-Grothendieck pour les algébres affinoïdes. Cela nous permet de déduire une version plus forte de la conjecture d'annulation analytique rigide d'Artin-Grothendieck sur un corps de caractéristique 0. En utilisant un ensemble complètement différent d'idées, nous obtenons également une version plus faible de cette conjecture sur un corps de caractéristique $p > 0$.

1. Introduction

1.1. Partial algebraization

In [4], M. Artin pioneered the study of algebraization questions in algebraic geometry. Although algebraization problems may take different forms depending on the exact problem under consideration, usually they can be summarized by the following (imprecise) question:

QUESTION 1.1. – Let A be a ring with a finitely generated ideal I, and let $\overline{S}$ be an "algebraic structure" over $A^{\wedge}_{I}$. Can we "algebraize" (or at least "approximate") it to an "algebraic structure" S over A?

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One positive answer to Question 1.1 is provided by [4, Th. 1.10] and its later generalization due to Popescu (see [54, Tag 0AH5]). It says that, for a noetherian G-ring A which is henselian along an ideal I, any solution $\widehat{y} \in (A^{\wedge}_I)^m$ of a (finite) system of polynomial equations $F_i \in A[X_1, \dots, X_m]$ can be approximated by a solution $y \in A^m$.

This result has become a standard (but very important) technical tool in many areas of algebraic geometry: various algebraization questions (see [4, Cor. 2.6 and § 3]), structure theory of algebraic stacks (see [3]), homological and commutative algebra (see [37]), construction of moduli spaces (see [5], [2]), and étale cohomology (see [42]).

Using the above approximation (and its version due to Elkik), we prove the following general algebraization result:

Theorem 1.2 (Noetherian rig-smooth algebraization; Theorem 5.15).

Let A be a noetherian ring, and let $I \subset A$ be an ideal. Let B be an I-adically complete A-algebra such that B is rig-smooth over (A, I) (in the sense of [54, Tag 0GAI]*) and B/IB is a finite type A/I-algebra. Then there is a finite type A-algebra C such that C is smooth outside $\mathrm{V}(I)$ and there is an isomorphism $C^{\wedge}_I \simeq B$ of A-algebras.*

Theorem 1.2 positively answers the question raised in [54, Tag 0GAX]. Of course, many special cases of Theorem 1.2 were known before. For instance, the case when A is a noetherian G-ring was settled in [54, Tag 0GAT], and the case when I is a principal ideal was settled in [25, Th. 7 on p. 592].

Theorem 1.2 provides a general tool for studying rig-smooth formal schemes. For instance, a version of Theorem 1.2 for rig-étale morphisms is used in the generalization of Artin's Theorem on dilatations (see [54, Tag 0ARB] and [54, Tag 0GDU], see also [6, Th. 3.2] for the original result), while Artin's Theorem on dilatations is used in the proof of Artin's Theorem on contractions (see [54, Tag 0GH7] and [6, Th. 3.1]).

However, the noetherian assumption in Theorem 1.2 is too limiting for the purposes of rigid-analytic geometry. Namely, one would like to apply this result to $A = \mathcal{O}_C$ for a rank-1 valuation ring in an algebraically closed non-archimedean field C. This ring is never noetherian, so Theorem 1.2 does not apply in this situation. Nevertheless, this issue has been overcome by R. Elkik, who studied an analogue of Artin approximation in certain non-noetherian situations. In particular, she proved the following remarkable fact:

Theorem 1.3 (Elkik's algebraization; special case of [25, Th. 7 on p. 592 and Rmq. 2(c) on p.598]).

Let K be a non-archimedean field with a pseudo-uniformizer $\varpi \in \mathcal{O}_K$, and let A be a rig-smooth(1)*, flat, topologically finite type $\mathcal{O}_K$-algebra. Then there is a flat, finite type $\mathcal{O}_K$-algebra B such that B_K is K-smooth and there exists an isomorphism $B^{\wedge}_{(\varpi)} \simeq A$ of $\mathcal{O}_K$-algebras.*

(1) See [51, Prop. 3.3.2] for the fact that a topologically finitely presented map $\mathcal{O}_K \to A$ is formally smooth "outside $\mathrm{V}(\varpi)$" in the sense of [25, p.591] if and only if $\mathcal{O}_K \to A$ is rig-smooth in the sense of [19, Def. 3.1]. Strictly speaking, [51, Prop. 3.3.2] is written under the additional assumption that K is discretely valued, but the proof goes through for a general non-archimedean field K.

Theorem 1.3 provides a very general machinery to reduce certain (local) questions about smooth rigid-analytic spaces to analogous questions about smooth algebraic varieties. For example, this approach has been used in [11] to study finiteness of étale cohomology of rigid-analytic spaces, in [52] to study questions related to semi-stable reduction, and in [57, § 2.5] to study properties of dualizing modules on rig-smooth admissible formal schemes.

However, Theorem 1.3 is still not strong enough to reduce questions about *singular* rigid-analytic spaces to analogous questions about singular algebraic varieties. Furthermore, the naive analogue of Theorem 1.3 is false if one does not impose any smoothness assumptions on A (see Appendix B.2). One of the key results of this paper is that it is nevertheless possible to algebraize a geometrically reduced affinoid rigid-analytic space in "one direction" under a very mild assumption.

Before we formulate the precise result, we recall that $\mathrm{Spa}\,(A, A^\circ)$ stands for the adic spectrum in the sense of Huber, see [38, § 3]. Furthermore, for a K-affinoid algebra A, the adic space $X = \mathrm{Spa}\,(A, A^\circ)$ is called *geometrically reduced* if $X \times_{\mathrm{Spa}\,(K, \mathcal{O}_K)} \mathrm{Spa}\,(L, \mathcal{O}_L)$ is reduced for any extension of non-archimedean fields $K \subset L$; see [23, § 3.3] for details. We warn the reader that [23, p. 509] provides an example of a non-archimedean field K and a geometrically reduced (in the usual algebraic sense, see [54, Tag 030S]) K-affinoid algebra A such that $\mathrm{Spa}\,(A, A^\circ)$ is *not* geometrically reduced.

Theorem 1.4 (Partial algebraization; Corollary 2.15). – *Let K be a non-archimedean field, let $\varpi \in \mathcal{O}_K$ be a pseudo-uniformizer, and let A_0 be a flat, topologically finite type $\mathcal{O}_K$-algebra such that $A := A_0[\frac{1}{\varpi}]$ is a K-affinoid algebra of Krull dimension $d > 0$ and* $\mathrm{Spa}\,(A, A^\circ)$ *is geometrically reduced. Put $R = \mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d]$. Then there is a finite, finitely presented $R^{\mathrm{h}}_{(\varpi)}$-algebra B with an isomorphism $B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)} \simeq A_0$ of $\mathcal{O}_K$-algebras.*

Corollary 1.5 (Partial algebraization II; Corollary 2.16). – *Let K, ϖ, and A_0 be as in Theorem 1.4 above. Then there is a finitely presented, quasi-finite morphism $\mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d] \to B$ and an isomorphism $B^{\wedge}_{(\varpi)} \simeq A_0$ of $\mathcal{O}_K$-algebras.*

We note that Theorem 1.4 and Corollary 1.5 are optimal. Namely, the assumption that $X = \mathrm{Spa}\,(A, A^\circ)$ is geometrically reduced in the sense of [23, p. 509] cannot be dropped in Theorem 1.4 and Corollary 1.5 (see also Remark 2.17). We refer to Proposition B.16 and Proposition B.21 for explicit counterexamples.

Unlike Theorem 1.3, Theorem 1.4 does not allow to directly reduce questions about geometrically reduced (affinoid) rigid-analytic spaces to similar questions about general (affine) algebraic varieties. However, it does give a quite robust tool of reducing problems about geometrically reduced (affinoid) rigid-analytic spaces of dimension d to algebro-geometric problems for (algebraic) curves over $\mathrm{Spec}\, K\langle T_1, \dots, T_{d-1}\rangle$. We implement this strategy to prove the remaining cases of Artin-Grothendieck vanishing for affinoid algebras, we discuss this proof in more detail in the next section. We also expect this strategy to be useful for other problems in rigid-analytic geometry. For example, the first author knows how to use Corollary 1.5 to prove [56, Theorem 1.3.5] without using any input from [15] or [49].