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THE WEAK BEAUVILLE-BOGOMOLOV DECOMPOSITION IN CHARACTERISTIC $p \geq 0$

BY ZSOLT PATAKFALVI AND MACIEJ ZDANOWICZ
WITH AN APPENDIX JOINT WITH GIULIO CODOGNI

ABSTRACT. – We prove a variant of the Beauville-Bogomolov decomposition for weakly ordinary, or generally globally F-split, varieties X with $K_X \sim 0$, in characteristic $p > 0$. We also show that the weakly ordinary assumption in our statement cannot be dropped. Additionally, if the assumption $K_X \sim 0$ is replaced by $-K_X$ being semi-ample, we show the weaker statement that all closed fibers of the Albanese morphism are isomorphic. Finally, we apply our main theorem to draw consequences to the behavior of rational points and fundamental groups of weakly ordinary K-trivial varieties in positive characteristic.

RÉSUMÉ. – On démontre un variant de la décomposition Beauville-Bogomolov pour les variétés faiblement ordinaire, ou F-scindé en général, avec $K_S \sim 0$ en caractéristic $p > 0$. On démontre aussi que l'hypothèse faiblement ordinaire est nécessaire. De plus, si l'hypothèse $K_X \sim 0$ est remplacée par le fait que $-K_X$ soit semi-ample, alors on démontre l'affirmation moins forte que toutes fibres du morphisme Albanese sont isomorphes. À la fin, on applique notre thórème principal pour arriver aux conséquences sur les points rationnels et les groupes foundamentaux des variétés faiblement ordinaires K-triviales en caractéristique positive.

1. Introduction

1.1. Historical motivation

The classical Beauville-Bogomolov decomposition (see [9, 7]) states that up to an étale cover, a compact Kähler manifold can be split into a product of an abelian variety, a Calabi-Yau variety U (satisfying $H^i(U, \mathscr{O}_U) = 0$, for $0 < i < \dim U$) and a hyperkähler manifold W (simply connected and admitting a holomorphic symplectic form).

The main goal of this article is to investigate what one can say about the existence of such decomposition over fields of positive characteristic. The above decomposition is based on the existence of Kähler-Einstein metrics, an analytic tool whose algebraic substitute is not known even in charateristic zero. Hence, in this article we analyze the weaker question that is approachable by algebraic tools, and to which we refer to as *weak Beauville-Bogomolov*

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decomposition. Over $\mathbb{C}$, it states that for every smooth projective variety X with $K_X \sim 0$ there exists a finite étale cover

$$V \times B \to X, \tag{1.0.a}$$

where B is an abelian variety and V is a smooth projective simply connected variety with $K_V \sim 0$. Already this statement has substantial corollaries concerning geometry of varieties with trivial canonical class. For example, it directly implies that the fundamental group of such a variety is virtually abelian, that is, admits a finite index abelian subgroup. We note that the weak Beauville-Bogomolov was already observed by Calabi for Kähler manifolds in [12] (see also [59]).

Additionally, instead of requiring V to be simply connected, the decomposition Equation 1.0.a can be alternatively also characterized up to an étale cover of V by using the notion of *augmented irregularity*. By definition, the augmented irregularity of X is

$$\widehat{q}(X) := \max\left\{\dim \mathrm{Alb}_{X'} \mid X' \to X \text{ is a finite étale morphism}\right\}, \tag{1.0.b}$$

for which one needs to prove that the above maximum exists. This follows from Calabi's original statement, or from [42, Thm 1] in a more general singular setting. Then, the above mentioned alternative characterization of Equation 1.0.a is by requiring that $\widehat{q}(X) = \dim B$ and $\widehat{q}(V) = 0$.

1.2. List of the results

In this subsection X is a smooth projective variety of dimension n over the stated base-field k. If we work over a perfect field of charateristic $p > 0$ and $K_X \sim 0$, then the only cohomology group that is guaranteed to be non-zero is $H^n(X, \mathscr{O}_X) \cong k$. In particular, the only arithmetic invariant one can extract is whether the Frobenius morphism acts on this cohomology as a bijection or not. We say that X is *weakly ordinary* if this action is bijective. Our results are:

- We fully answer the question when a weak Beauville-Bogomolov decomposition exists. That is,
  1. In the weakly ordinary case we clarify the relaxations of the statement over $\mathbb{C}$ (see § 1.1) needed to obtain a weak Beauville-Bogomolov decomposition:
     - (a) V has to be allowed mildly singular, and
     - (b) the morphism $V \times B \to X$ has to be allowed to be inseparable, i.e., a torsor under a possibly non-reduced group scheme.

     (Theorem 1.1 is the smooth case, and Theorem 11.6 is the singular and pair case.)

     Additionally, we note that the above two relaxations are unavoidable (see [49, Thm 2.11] and Example 13.5).
  2. If X is not weakly ordinary, then we find an example that cannot admit a decomposition (Proposition 14.3).
- We deduce corollaries to the number of rational points and to the behavior of the étale fundamental group (see Corollary 12.5 and Corollary 12.9).

— We solve the positive characteristic version of the Demailly-Peternell-Schneider conjecture explained in the following paragraph (Theorem 1.7 is the smooth case, and Theorem 9.3 is the singular and pair case). This proof is a byproduct of our methods for solving the above Beauville-Bogomolov question, and it requires practically no additional effort.

1.3. The smooth cases of the characteristic $p > 0$ results

Our base field k is perfect and of characteristic $p > 0$, and we also need the following positive characteristic notion: a projective variety X over k is *weakly ordinary* if the action of the absolute Frobenius morphism F_X on $H^{\dim X}(X, \mathcal{O}_X)$ is bijective. This is a genericity notion. That is, being weakly ordinary is an open condition in positive equicharacteristic and it is typically dense in moduli. Additionally, it is conjectured to be dense over mixed characteristic bases that are finite type over $\mathbb{Z}$ [66]. For a smooth projective weakly ordinary variety X with $K_X \sim 0$ we define the *augmented irregularity* just as in Equation 1.0.b, where the existence of the maximum is guaranteed by [23, Thm 1.1].

Also, already when X is smooth, our decomposition statement uses the notion of strongly F-regular singularities, which is a characteristic p class of mild singularities. In particular, it is contained both in the class of klt and rational singularities. We refer to the many surveys in the topic for a detailed introduction on the notions of F-singularities [82, 73, 72]

THEOREM 1.1 (Smooth & weakly ordinary case of the Beauville-Bogomolov decomposition, special case of Theorem 11.6).

Let X be smooth projective variety over k, such that $K_X \sim 0$ and X is weakly ordinary. Then there is a composition

$$B \times V \to Z \to X$$

of two finite covers, such that

(1) *$Z \to X$ is étale, $B \times V \to Z$ is a torsor under $\prod_{i=1}^{\widehat{q}(X)} \mu_{p^{j_i}}$ for some integers $j_i \geq 0$,*

(2) *B is an abelian variety with* $\dim B = \widehat{q}(X)$*, and*

(3) *V is a weakly ordinary projective Gorenstein variety over k with strongly F-regular singularities, $K_V \sim 0$ and $\widehat{q}(V) = 0$.*

Additionally, the action of $G := \prod_{i=1}^{n} \mu_{p^{j_i}}$ on $B \times V$ is the diagonal action induced by an action on V and an action on B, respectively, such that

— *G acts freely and faithfully on B, and*

— *G acts faithfully on V.*

REMARK 1.2. – There are two major differences between Theorem 1.1 and the original characteristic zero statement mentioned above:

1. $B \times V \to X$ in Theorem 1.1 is not étale as in characteristic zero, but has an infinitesimal part as well, and
2. V is not necessarily smooth, but has only strongly F-regular singularities.