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TOPOLOGICAL ENTROPY OF A RATIONAL MAP OVER A COMPLETE METRIZED FIELD

BY Charles FAVRE, Tuyen Trung TRUONG and Junyi XIE

ABSTRACT. – We prove that the topological entropy of any dominant rational self-map of a projective variety defined over a complete non-Archimedean field is bounded from above by the maximum of its dynamical degrees. This extends a theorem of Gromov, Dinh and Sibony from the complex to the non-Archimedean setting. We proceed by proving that any regular self-map which admits a regular extension to a projective model defined over the valuation ring has necessarily zero entropy. To this end we introduce the ϵ-reduction of a Berkovich analytic space, a notion of independent interest.

RÉSUMÉ. – Nous démontrons que l'entropie topologique d'une application rationnelle dominante d'une variété projective définie sur un corps K complet non-archimédien est bornée supérieurement par le suprémum de ses degrés dynamiques, généralisant ainsi un théorème de Gromov, Dinh et Sibony du cas complexe au cas non-archimédien. Nous montrons de plus que tout endomorphisme qui admet une extension régulière sur un modèle défini sur l'anneau de valuation de K est d'entropie nulle. Pour ce faire, nous introduisons le concept de ϵ-réduction pour les espaces de Berkovich, une notion d'intérêt général en géométrie analytique non-archimédienne.

Introduction

The most basic dynamical invariant associated with a continuous self-map of a compact space is arguably its topological entropy, first defined by Adler-Konheim-McAndrew in [1]. In order to better understand the dynamics of maps of algebraic origin, we propose in this paper to estimate the topological entropy of rational self-maps of projective varieties defined over a non-Archimedean complete metrized field.

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Let us first consider a dominant rational self-map $f\colon X \dashrightarrow X$ of a complex projective normal variety X , i.e., a projective normal reduced and geometrically integral scheme over $\mathbb{C}$, of dimension $d \geq 1$. The complex analytic variety X^{an} is then compact and metrizable, and even though f may have indeterminacy points, one can define its topological entropy "à la Bowen" [13] by counting the exponential growth of the number of (n, ε)-separated well-defined orbits as $n \to \infty$. Gromov [35] proved that $\mathrm{h}_{\mathrm{top}}(f) \leq \log(\deg(f))$ for any regular endomorphism of the projective space where $\deg(f)$ denotes the topological degree of f [(1)]; later Dinh and Sibony [20, 21] partially extended this result to any meromorphic self-map of a compact Kähler manifold (so in particular to any rational self-map of a complex projective manifold). Indeed, they proved:

$$\mathrm{h}_{\mathrm{top}}(f) \leq \max_{0 \leq i \leq d} \log \lambda_i(f), \tag{1}$$

where $\lambda_i(f) \geq 1$ are the dynamical degrees of f (we discuss below their definitions in the context of projective varieties).

Now let $(k, |\cdot|)$ be a complete non-Archimedean field, and let $f\colon X \dashrightarrow X$ be a dominant rational self-map of a projective variety X defined over k. Defining the topological entropy of f raises some difficulties in this context, since the set $X(k)$ of k-points in X can be endowed with a metric topology coming from the norm on k, but this space is not locally compact if $d \geq 1$ except if k is a local field. To get around this problem, one considers the Berkovich analytification X^{an} of X, see [7]. This space has good topological properties and is in particular compact. The map f induces an analytic (hence continuous) map on a dense subset of X^{an}. Note that most of the time (e.g., when the residue field of k is uncountable) X^{an} is not metrizable, and the topological entropy of a continuous self-map of X^{an} is defined as the growth of the complexity of the sequence of open covers $\mathfrak{U} \vee \cdots \vee f^{-n}\mathfrak{U}$ where $\mathfrak{U}$ is a fixed open cover of X. This definition is convenient when f is regular, but it is not easy to adapt it when f is merely rational since it is not immediately related to orbits of points. We explain in detail in §1.3 how to interpret Adler-Konheim-McAndrew's original definition in a Bowen perspective using the canonical uniform structure on X^{an} (see also [43]). We show that the topological entropy is computed by the growth of the cardinality of maximal $(n, \mathcal{E})$-separated sets where $\mathcal{E}$ is any entourage of the diagonal in $X^{\mathrm{an}} \times X^{\mathrm{an}}$. Using this interpretation, it is then not difficult to define the topological entropy of any rational self-map, even though it only defines a partially continuous map on X^{an}, see §2.4.

To state our main theorem, we now need to discuss the notion of dynamical degrees first introduced by Russakovskii-Shiffman [57]. For any ample line bundle $L \to X$ and for any integer $j \in \{0, \ldots, d\}$, set $\deg_{L,j}(f) = f^* c_1(L)^j \cdot c_1(L)^{d-j}$. When $k = \mathbb{C}$ or more generally if the characteristic of k is zero, then it is a theorem of Dinh and Sibony [20] that the sequence $\{C \deg_{L,j}(f^n)\}_{n \in \mathbb{N}}$ is sub-multiplicative for some $C > 0$, so that one can define the j-th dynamical degree of f by setting $\lambda_j(f) := \lim_n \deg_{L,j}(f^n)^{1/n}$. Dynamical degrees do not depend on the choice of L, and are invariant under birational conjugacies. When the characteristic of k is positive, the sub-multiplicativity of the sequence of degrees (hence the

(1) In fact the equality $\mathrm{h}_{\mathrm{top}}(f) = \log(\deg(f))$ holds. The lower bound was proved earlier by Misiurewicz-Przytycki and a far reaching generalization was then obtained by Yomdin, see the discussion on the next page.

existence of the dynamical degrees) was first obtained by the second author [62], and an alternative approach has been subsequently developped by N.-B. Dang [15].

Our first result states that the upper bound (1) is true over any field.

THEOREM A. – *Let $(k, |\cdot|)$ be any complete non-Archimedean field, and let $f\colon X \dashrightarrow X$ be any dominant rational self-map of a projective variety X defined over k.*

Then the topological entropy of the map induced by f on the Berkovich analytification of X is bounded from above by $\max_{0\le i\le \dim(X)} \log \lambda_i(f)$.

This result was obtained by the first author and Rivera-Letelier in the case $X = \mathbb{P}^1_k$, see [26, Théorème C].

Let us briefly indicate how our proof works in arbitrary dimension. For each n, consider the Zariski closure $\Gamma_n \subset X^n$ of the set of orbits of length n, that is of the set of points $(x, \dots, f^{n-1}(x))$ where x is not indeterminate under $f, \dots, f^{n-1}$. Observe that Γ_n is an algebraic variety of dimension $d := \dim(X)$ for all n. Fix an ample line bundle $L \to X$, and let L_n be the restriction to Γ_n of the line bundle $\sum_{i=0}^{n-1} \pi_i^* L$ where π_i is the projection $X^n \to X$ onto the i-th factor, and where we use additive notation for the tensor product of line bundles.

In the complex setting, we may choose a smooth positive metrization on L whose curvature defines a Kähler form ω on X. By Wirtinger's theorem the volume of Γ_n for the Kähler form $\omega_n := \sum_{i=0}^{n-1} \pi_i^* \omega$ can be computed in cohomological terms, and we get $\mathrm{Vol}_{\omega_n}(\Gamma_n) = \deg_{L_n}(\Gamma_n) := c_1(L_n)^{\wedge d}$. By a clever calculation due to Gromov, we obtain $\limsup_n \frac{1}{n} \log \deg_L(\Gamma_n) \le \max_{0\le i\le d} \log \lambda_i(f)$. On the other hand, a theorem by Lelong implies the volume $\mathrm{Vol}_{\omega_n}(\Gamma_n)$ to be no less than the number of (n, ε)-separated orbits up to a uniform constant which implies (1).

Definitions of a Kähler form have been proposed in [11, §2.3] and [68, §3] in the non-Archimedean case using the notion of models over the ring of integers $k^\circ = \{|z| \le 1\} \subset k$. The precise definition of a Kähler form is not relevant to our discussion, but models play a crucial role in our approach. A model of X over k° will be by convention a flat projective scheme $\mathfrak{X} \to \operatorname{Spec} k^\circ$ whose generic fiber is isomorphic to X. In our argument, the choice of a relatively ample line bundle $\mathfrak{L} \to \mathfrak{X}$ replaces the choice of a Kähler form in the complex setting.

Given any entourage $\mathcal{E}$ of the diagonal, we are first able to construct a suitable model $\mathfrak{X}$ of X such that for any integer $n \ge 0$ the following holds. The number of $(n, \mathcal{E})$-separated points is bounded from above by the number Q_n of irreducible components of the central fiber of the model $\mathfrak{G}^{\langle n\rangle}$ of Γ_n obtained by taking the closure of Γ_n inside $\mathfrak{X}^n$. Next we prove that

$$Q_n \le c_1(\mathfrak{L}_n)^{\wedge d} \cdot \mathfrak{G}_s^{\langle n\rangle},$$

where $\mathfrak{L} \to \mathfrak{X}$ is a relatively ample line bundle, $\mathfrak{L}_n = \sum_{i=0}^{n-1} \pi_i^* \mathfrak{L}$, and π_i is the projection $\mathfrak{X}^n \to \mathfrak{X}$ onto the i-th factor. Using the invariance of intersection numbers under flat morphism, we get $Q_n \le c_1(L_n)^{\wedge d}$. We then adapt Gromov's calculation using ideas from [15] to relate $c_1(L_n)^{\wedge d}$ to the dynamical degrees, and our proof is complete. □

In the complex case, it follows from a deep theorem of Yomdin [67, 34] that Gromov's upper bound is attained whenever f is regular. The case of rational maps has triggered