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ACTIONS OF TENSOR CATEGORIES ON KIRCHBERG ALGEBRAS

BY Kan KITAMURA

Abstract. – We characterize the simplicity of Pimsner algebras for non-proper C*-correspondences. With the aid of this criterion, we give a systematic strategy to produce outer actions of unitary tensor categories on Kirchberg algebras. In particular, every countable unitary tensor category admits an outer action on the Cuntz algebra $\mathcal{O}_2$. We also study the realizability of modules over fusion rings as K-groups of Kirchberg algebras acted on by unitary tensor categories, which turns out to be generically true for every unitary fusion category. Several new examples are provided, among which actions on Cuntz algebras of 3-cocycle twists of cyclic groups are constructed for all possible 3-cohomological classes, thereby answering a question asked by Izumi.

Résumé. – Nous caractérisons la simplicité des algèbres de Pimsner pour les C*-correspondances non propres. En utilisant ce critère, nous proposons une stratégie systématique pour produire des actions extérieures de catégories tensorielles unitaires sur les algèbres de Kirchberg. En particulier, chaque catégorie tensorielle unitaire dénombrable admet une action extérieure sur l'algèbre de Cuntz $\mathcal{O}_2$. Nous étudions également la réalisabilité de modules sur des anneaux de fusion en tant que K-groupes d'algèbres de Kirchberg soumises à des actions de catégories tensorielles unitaires, ce qui s'avère génériquement possible pour chaque catégorie de fusion unitaire. Plusieurs nouveaux exemples sont obtenus, parmi lesquels des actions sur les algèbres de Cuntz par des twists de 3-cocycles de groupes cycliques sont construites pour toutes les classes de 3-cohomologie possibles, ce qui résout une question posée par Izumi.

1. Introduction

In the theory of subfactors, initiated by Jones [39], it was found that inclusions of operator algebras can give rise to new types of symmetries beyond groups, sometimes called quantum symmetries. This discovery led to unexpected connections between operator algebras and other branches of mathematics and physics, such as low-dimensional topology and quantum field theory. As concrete realizations of quantum symmetries, the construction and classification of subfactors have been attractive themes of operator algebras. Beyond Ocneanu's classification [60] of actions of amenable discrete groups on AFD II_1 factors, a celebrated

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result of Popa [66, 67] shows that amenable inclusions of AFD II_1 factors are classified by their standard invariants. One of the equivalent ways to formulate this standard invariant uses the notions of actions of tensor categories and Q-systems [51, 54].

Recently, Gabe-Szabó [25] made a breakthrough in the classification of amenable actions of groups on purely infinite C*-algebras. Their main result is the equivariant generalization of the Kirchberg-Phillips classification theorem [46, 64], stating that amenable actions of groups on Kirchberg algebras are classified by their equivariant KK-theory. Since then, there has been increasing attention to the classification of actions of tensor categories on C*-algebras. For example, see [14] in the approximately finite setting and [3, 27] for some further attempts.

In the C*-algebraic case, new obstructions to the existence of certain quantum symmetries have been observed in [38, 22, 36]. These obstructions have cohomological or K-theoretic origins, suggesting that C*-algebras can reflect different subtleties of quantum symmetries from those appearing in subfactor theory. There are several known constructions of actions of tensor categories on simple nuclear C*-algebras [31, 32, 21] besides those on AF algebras obtained from Ocneanu's compactness argument [61]. However, these constructions often impose severe restrictions on the shapes of resulting C*-algebras, and not so much has been known about the existence of quantum symmetries on given C*-algebras.

In this paper, we initiate a systematic study of their existence on C*-algebras in the purely infinite case. We establish a new strategy to construct actions of tensor categories on Kirchberg algebras based on the following machinery.

THEOREM 1.1 (Theorem 4.9). – *For a given action of a countable unitary tensor category $\mathcal{C}$ on a separable nuclear C*-algebra A, there is an outer $\mathcal{C}$-action on a unital Kirchberg algebra B with a good control of* K*-theoretic data: more precisely, B is $\mathcal{C}$-equivariantly* KK*-equivalent to A in the sense of* [3].

Thanks to the Kirchberg-Phillips theorem [46, 64], this machinery allows us to resort to topological or homologically algebraic methods to obtain plenty of quantum symmetries on given Kirchberg algebras. We first describe a few consequences for C*-algebraic analogues of questions in subfactor theory.

In his seminal work [39], Jones proved that the indices of inclusions of II_1 factors $N \subset M$ must be contained in $\{4\cos^2\frac{\pi}{k+2} \mid k \in \mathbb{N}\} \cup [4, \infty]$, and all of these values can be realized by some $N \subset M$. Moreover, M and N can be taken as AFD factors satisfying $N' \cap M = \mathbb{C}1_M$ when the index is of the form $4\cos^2\frac{\pi}{k+2}$. In addition, since [39, Problem 1], it is a long-standing open problem which indices in $(4, \infty)$ can be realized in this way.

More generally, it is natural to ask which tensor categories $\mathcal{C}$ admit an outer action on some AFD factor [10, Question 1.2], [69, 6.3.1]. (Here, note that any outer $\mathcal{C}$-action on an AFD factor gives rise to some outer $\mathcal{C}$-action on the AFD II_1 factor by tensoring with the AFD type III_1 factor and then passing to its continuous core using [34], which is of type II_∞ and thus Moria equivalent to the AFD II_1 factor.) A deep result of Popa and Shlyakhtenko [68, 70] shows that finite index inclusions of $L(\mathbb{F}_\infty)$ can give rise to arbitrary finitely generated tensor categories $\mathcal{C}$. Nevertheless, when it comes to AFD factors, this question is already open even when $\mathcal{C}$ is the representation category of $\mathrm{SU}_q(2)$ for $0 < q < 1$, whose affirmative answer would immediately imply the affirmative answer to [39, Problem

1] above at the index $(q+q^{-1})^2 \in (4,\infty)$. We consider the analogues of these problems for simple nuclear C*-algebras, which turn out to have affirmative answers.

THEOREM 1.2 (Corollary 4.12, Proposition 5.9). – *The following hold.*

1. *Every countable unitary tensor category admits an outer action on the Cuntz algebra* $\mathcal{O}_2$.
2. *For any unital Kirchberg algebra* A*, every value in* $[4,\infty)$ *can be realized as the Watatani index of some unital* $*$*-endomorphism* $\iota\colon A \to A$ *with* $\iota(A)' \cap A = \mathbb{C}1_A$.

In (2), we can choose the larger and smaller C*-algebras to be $*$-isomorphic because of the particular flexibility of the K-theoretic data due to the homological well-behavior of the representation ring of $\mathrm{SU}_q(2)$. However, we cannot hope that (2) holds unconditionally when the index is smaller than 4. This is because the quantum symmetries arising from the inclusions of such indices are closely related to the unitary fusion category of SU(2) at level $k \in \mathbb{N}$, denoted by $\mathcal{C}(\mathfrak{sl}_2,k)$, and its fusion ring is homologically more complicated than the representation ring of $\mathrm{SU}_q(2)$. This situation imposes some K-theoretic restriction on A for (2) in this case. Similarly, (1) is no longer valid for arbitrary Kirchberg algebras due to K-theoretic obstructions. To understand these situations, we next investigate the possible K-theoretic data coming from actions of fusion categories.

The first constraint for a tensor category $\mathcal{C}$ to act on a C*-algebra A is that it must induce $\mathbb{Z}[\mathcal{C}]$-module structures on $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$. Conversely, it is natural to ask when the existence of $\mathbb{Z}[\mathcal{C}]$-module structures on the K-groups assures the existence of $\mathcal{C}$-actions on A or, more elaborately, when given $\mathbb{Z}[\mathcal{C}]$-module structures on them lift to some $\mathcal{C}$-action on A. Affirmative answers to this question will provide sufficient conditions in algebraic terms of K-theory for the existence of $\mathcal{C}$-actions on A, which is an analytic situation. A similar question has been considered in the group case [73, 44], and Katsura [8] shows that this is always true for certain finite groups including all finite cyclic groups. Unfortunately, typically $\mathbb{Z}[\mathcal{C}]$-modules do not lift to $\mathcal{C}$-actions on Kirchberg algebras, even when $\mathcal{C}$ is a non-trivial 3-cocycle twist of a finite group [26]. Yet, using homologically algebraic methods with the aid of Theorem 1.1, we can still show that this liftability is generically true for arbitrary fusion categories.

THEOREM 1.3 (Corollary 5.11). – *For every unitary fusion category* $\mathcal{C}$*, there exists* $d \in \mathbb{N}$ *such that, for all countable* $\mathbb{Z}[\mathcal{C}]$*-modules* M_0 *and* M_1 *on which* d *acts invertibly, there is a Kirchberg algebra* A *in the UCT class acted on by* $\mathcal{C}$ *satisfying* $M_0 \cong \mathrm{K}_0(A)$ *and* $M_1 \cong \mathrm{K}_1(A)$ *as* $\mathbb{Z}[\mathcal{C}]$*-modules.*

We will also give a computable choice of this constant $d \in \mathbb{N}$. However, this choice is often far from optimal, and determining the smallest value of d is much more difficult in general. What we can do at present is the optimization of such d when the fusion category is $\mathcal{C}(\mathfrak{sl}_2,k)$ or its variants for a certain series of k.

THEOREM 1.4 (Theorem 6.3, Example 7.6). – *Let* p *be an odd prime. Then, we can take* $d=1$ *in Theorem 1.3 when* $\mathcal{C}$ *is the even part of* $\mathcal{C}(\mathfrak{sl}_2,p-2)$ *or the non-trivial twist of* $\mathcal{C}(\mathfrak{sl}_2,p-2)$*. Also, we can take* $d=2$ *(but cannot take* $d=1$*) when* $\mathcal{C}$ *is* $\mathcal{C}(\mathfrak{sl}_2,p-2)$ *itself.*