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HILBERT'S 6TH PROBLEM AND BOHLMANN'S "GRUNDBEGRIFFE"

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Abstract. — We discuss the emergence of Hilbert's 6th problem and Hilbert's presentation of Bohlmann's axioms. An improved version of the axioms, which appeared in 1908, has been largely ignored by history. It overlaps with Kolmogorov's "Grundbegriffe".

Résumé (Le sixième problème d'Hilbert et les « Grundbegriffe » de Bohlmann)
Nous discutons de l'émergence du 6e problème de Hilbert et de la présentation par Hilbert des axiomes de Bohlmann. Une version améliorée des axiomes, parue en 1908, a été largement ignorée par l'histoire. Elle recoupe les « Grundbegriffe » de Kolmogorov.

In his talk at the International Mathematics Conference in Paris (1900), Hilbert had proposed as his sixth problem the axiomatization of probability, along the lines of his axiomatic approach to geometry that had changed the traditional role of axioms, [Hilbert 1900]. Kolmogorov's axiomatization from 1933 [Kolmogorov 1933] is classically seen as the answer to that challenge, raising the standard of rigor for probability. In this paper we

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want to show that Georg Bohlmann played an important part in the beginning of these events and present new arguments in favor of the assertion that the probability part of Hilbert's 6th problem was suggested by Bohlmann.

Some years later, in 1905, Hilbert presented the axioms proposed by Bohlmann in one of his lectures. He made it clear that they did not have the final form, but the fact that he presented them in a lecture shows that he thought that Bohlmann was on the right track. We reproduce Max Born's handwritten notes of Hilbert's presentation of the Bohlmann axioms. Bohlmann presented an improved version at the Fourth World Congress of Mathematicians in Rome in 1908. We shall see that it anticipates some of the contributions of Kolmogorov.

1. THE EMERGENCE OF HILBERT'S 6TH PROBLEM

The 19th century had led to important progress in proving limit theorems in probability theory, but the discussion of the basic concepts had remained controversial. This can be shown, e.g., by an example from the textbook of A. A. Markov [Markoff 1912, p. 5–6]. An urn contains balls with four different colors 1, 2, 3, 4 with unknown frequencies a, b, c, d. We call it Markov's urn. Markov asserts, that the probability of drawing a ball with the color 1 is $\frac{1}{4}$, because all colors are equally possible. He then continues: If we are given the information that $a < b < c < d$ holds, we should abstain from making assumptions about the probabilities. If Markov is right, lack of knowledge has helped to determine a probability. In this example, Markov has not distinguished two kinds of uncertainty. In an urn with known parameters a, b, c, d the probability of drawing a ball with the color 1 is $a/(a+b+c+d)$. If we have the added uncertainty about the values of the parameters, the probability of drawing a ball with the color 1 remains the same, but we do not know it. It is not necessarily $\frac{1}{4}$. One can try to find an approximation by sampling with replacement.

In the 19th century, the discussion whether lack of knowledge can help to determine a probability was still not over. What was missing most was the mathematical meaning of the term probability. This situation certainly has been the main reason for the formulation of the sixth of Hilbert's problems. Hilbert posed 23 problems at the International Congress of Mathematicians in Paris on August 8, 1900. We begin by stating a translation of the relevant part of Hilbert's sixth problem with the original wording [Hilbert 1900].

> **Mathematical Treatment of the Axioms of Physics.** The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probability and mechanics. As to the axioms of the theory of probability (cf. Bohlmann, "Ueber Versicherungsmathematik," 2nd lecture in Klein und Riecke, *Ueber angewandte Mathematik und Physik*, Leipzig und Berlin 1900), it seems to me desirable that their logical investigation should be accompanied by a rigorous and satisfactory development of the method of mean values in mathematical physics, and in particular in the kinetic theory of gazes..

In this formulation, Hilbert quotes Bohlmann's paper [Bohlmann 1900], but this quotation was omitted in more recent republications of his talk.

Studying the quoted paper, I was surprised to find out that it is completely unrelated to Hilbert's talk up to a footnote quoting the paper [Bohlmann 1901]. It deals with lectures by Bohlmann on how to teach actuarial mathematics in high schools. These lectures were given by Bohlmann during the Easter vacation of 1900 to an audience of high school teachers.

But, in addition to the paper [Bohlmann 1900] Bohlmann had written an article [Bohlmann 1901] for the *Enzyklopädie der mathematischen Wissenschaften*. In this article Bohlmann stated that the mathematics of life insurance has to be built on the theory of probability, and he started by formulating axioms for probability theory. In a footnote of [Bohlmann 1900], Bohlmann had remarked: "A more detailed list of the axiomatic structure given here can be found in my paper for the mathematical encyclopedia volume 1, article D.4.c)." Thus, the complete reference for the encyclopedic article was available by Easter 1900 (but not yet printed), and Hilbert preferred to quote the Easter lecture. For Hilbert, a reason to quote the Easter lectures and not the encyclopedic article may have been that he wanted to acknowledge that Bohlmann had done his work before Hilbert started to think about his list of problems. Indeed, the encyclopedic article only appeared after the Hilbert talk. I conclude: The quoted paper was unimportant to Hilbert, but, apparently, it was important to him to quote Bohlmann.

I must explain some of the background. Georg Bohlmann was a mathematician, who had studied in Berlin. He obtained his doctoral degree in Halle in 1892. Felix Klein had asked Bohlmann to come to Göttingen, where he obtained the habilitation (permission to teach) in 1894. He had given lectures on actuarial mathematics since 1895. In 1899, Klein had

proposed to the ministry in Berlin that Bohlmann should be granted the title of extraordinary professor. This application was accepted in 1901.

Bohlmann's axioms in [Bohlmann 1901] were inspired by Hilbert's approach to geometry. Hilbert had lectured on the foundations of geometry in the winter term 1898-1899. For him, axioms are no longer regarded as directly recognizable truths from which other truths are derived by logical inference. They were regarded as starting points for a theory. Bohlmann was familiar with Hilbert's new ideas.

When Hilbert was invited to give a plenary lecture to the Second International Congress of Mathematicians, scheduled for Paris 1900, he consulted his friends Minkowski and Hurwitz. This correspondence is discussed in [Gray 2000, pp. 57–59]. Minkowski had suggested the look into the future of mathematics. Hilbert did a poor job of keeping in touch with the organizers of the Conference, and in June 1900, the program of the meeting was mailed to those who had indicated interest without Hilbert's name being listed. In a letter to Hilbert dated June 22, 1900, Minkowski wrote that he was very disappointed and therefore hesitated to go to Paris at all [Rüdenberg & Zassenhaus 1973, p. 126].

Hilbert made no secret about his asking unnamed colleagues which problems seemed important to them. His point was not that these were his problems, but that they might be important in the next century.

All this, together with the fact that apparently Hilbert felt that he should quote Bohlmann, leaves little doubt that Hilbert and Bohlmann had talked to each other, and Bohlmann had explained which problem occupied his mind. Hilbert thought this was a good problem, and it was clear, that the axioms of Bohlmann were not sufficient. Hilbert saw that enough work remained to be done, and decided to include the axiomatization in his list of problems.

A crucial argument supporting this view: There is not the slightest hint that Hilbert had thought about axioms for probability theory before he knew that Bohlman worked in this direction. Bohlmann would have been proud to mention that he worked on a problem suggested by Hilbert.

Hilbert did his best to acknowledge Bohlmann's contribution and certainly Bohlmann's project was ennobled by its inclusion in Hilbert's list of problems.

2. HILBERT'S PRESENTATION OF BOHLMANN'S EARLY AXIOMS

In the summer of 1905, Hilbert gave a course on "Logical Principles of Mathematical Thinking" and in one of the lectures he gave a presentation

of Bohlmann's approach, which had been a start of an axiomatization in the spirit of Hilbert.

Max Born, later a famous physicist and Nobel laureate, took careful handwritten notes of Hilbert's lecture for his own use. He presented a volume with these notes to Friedrich Hund on the occasion of Hund's 70th birthday. Hund, also a famous quantum physicist, had been his doctoral student. Finally, Hund, who reached the age of 101, donated the volume to the Göttingen University Library.[1] As the Born notes are so far only available in the volume which Born gave to Hund, we now present a copy of these notes and a translation into English[2]. I should mention that a second set of lecture notes of the Hilbert lecture exists. It was written by Ernst Hellinger [Hellinger 1905]. It is also handwritten, and with regard to content it agrees with the Born notes.

I do not know, if Bohlmann ever saw or heard Hilbert's comments about his axioms, but he was aware of the fact that one of Hilbert's student, Broggi, later wrote a thesis about the axiomatization of probability.

2.1. *Translation of the Born notes*

Axioms of Probability

In the scientific literature, there has been a series of preliminary attempts to treat the calculus of probability axiomatically. The reason that these have not led to definitive analyses is that here, like everywhere, most scientists have abstained from considering several fields at the same time; yet it is precisely the comparison that is stimulating and fruitful. For example, through this kind of comparative description of axiomatizations, I have observed the following fact: A basic step in building each discipline seems to be finding the form of a certain function with the aid of a few axioms describing apparent, plausible, or empirically proven facts. For example, the axiomatization of geometry comes out of the determination that the equation of a straight line has the form of a linear equation. In thermodynamics, we have just seen that everything comes down to

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1 It is interesting to note that the opening page of the volume that Born gave to Hund contains a handwritten poem by Born. Here is a transcription of this poem:

Friedrich Hund zum siebzigsten Geburtstag
mit der Ausarbeitung einer Hilbert-Vorlesung:

Wer so was macht	Nach 60 Jahren	Nach solcher Frist	Der künftigen Welt
wird ausgelacht	wird er erfahren:	historisch ist	man's gern erhält.
und bald bekannt	die Zeit verkehrt	was Hilbert lehrte,	Des eingedenk
als ein Pedant.	der Dinge Wert.	der Hochverehrte.	hier mein Geschenk.

Max Born.

2 The scan of the notes of Max Born and the translation were also contained in the draft of an unpublished joint paper of U. Krengel and G. Shafer many years ago.