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# ON ARRANGEMENTS OF THE ROOTS OF A HYPERBOLIC POLYNOMIAL AND OF ONE OF ITS DERIVATIVES

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***Abstract.*** — We consider real monic *hyperbolic* polynomials in one real variable, *i.e.* polynomials having only real roots. Call *hyperbolicity domain* $\Pi$ of the family of polynomials $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$, the set $\{a \in \mathbf{R}^n \mid P \text{ is hyperbolic}\}$. The paper studies a stratification of $\Pi$ defined by the arrangement of the roots of $P$ and $P^{(k)}$, where $2 \leqslant k \leqslant n-1$. We prove that the strata are smooth contractible semi-algebraic sets.

***Résumé* (Sur les arrangements des racines d'un polynôme hyperbolique et d'une de ses dérivées)**

Nous considérons des polynômes moniques *hyperboliques* à une variable réelle, c'est-à-dire des polynômes dont toutes les racines sont réelles. Définissons le *domaine d'hyperbolicité* $\Pi$ de la famille de polynômes $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$, comme l'ensemble $\{a \in \mathbf{R}^n \mid P \text{ est hyperbolique}\}$. L'article étudie la stratification de $\Pi$ définie par l'arrangement des racines de $P$ et de $P^{(k)}$, où $2 \leqslant k \leqslant n-1$. Nous montrons que les strates sont des ensembles lisses, contractibles et semi-algébriques.

## 1. Introduction

In the present paper we consider real monic *hyperbolic* (resp. *strictly hyperbolic)* polynomials in one real variable, *i.e.* polynomials having only real (resp. only real distinct) roots. If a polynomial is (strictly) hyperbolic, then so are all its non-trivial derivatives.

Consider the family of polynomials $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$. Call *hyperbolicity domain* $\Pi$ the set $\{a \in \mathbf{R}^n \mid P \text{ is hyperbolic}\}$. The paper studies a stratification of $\Pi$ defined by the *configuration* (we write sometimes *arrangement*) of the roots of $P$ and $P^{(k)}$, where $2 \leqslant k \leqslant n-1$. The study of this stratification began in [**KoSh**], see also [**Ko1**] and [**Ko2**] for the particular cases $n = 4$ and $n = 5$.

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Properties of $\Pi$ were proved in [**Ko3**] and [**Ko4**], the latter two papers use results of V.I. Arnold (see [**Ar**]), A.B. Givental (see [**Gi**]) and I. Meguerditchian (see [**Me1**] and [**Me2**]).

***Notation 1***. — Denote by $x_1 \leqslant \cdots \leqslant x_n$ the roots of $P$ and by $\xi_1 \leqslant \cdots \leqslant \xi_{n-k}$ the ones of $P^{(k)}$. We write sometimes $x_i^{(k)}$ instead of $\xi_i$ if the index $k$ varies. Denote by $y_1 < \cdots < y_q$ the *distinct* roots of $P$ and by $m_1, \ldots, m_q$ their multiplicities (hence, $m_1 + \cdots + m_q = n$).

The classical Rolle theorem implies that one has the following chain of inequalities:

$$x_i \leqslant \xi_i \leqslant x_{i+k}, \quad i = 1, \ldots, n-k \tag{1}$$

***Definition 2***. — A *configuration vector (CV)* of *length* $n$ is a vector whose components are either positive integers (sometimes indexed by the letter $a$, their sum being $n$) or the letter $a$. The integers equal the multiplicities of the roots of $P$, the letters $a$ indicate the positions of the roots of $P^{(k)}$; $m_a$ means that a root of $P$ of multiplicity $m < k$ coincides with a simple root of $P^{(k)}$. A CV is called *a priori admissible* if inequalities (1) hold for the configuration of the roots of $P$ and $P^{(k)}$ defined by it.

***Remark 3***. — If a root of $P$ of multiplicity $< k$ is also a root of $P^{(k)}$, then it is a simple root of $P^{(k)}$, see Lemma 4.2 from [**KoSh**]. By definition "a root of multiplicity 0" means "a non-root".

***Example 4***. — For $n = 8$, $k = 3$ the CV $(1, a, 1, 2_a, a, a, 4)$ (which is a priori admissible) means that the roots $x_j$ and $\xi_i$ are situated as follows: $x_1 < \xi_1 < x_2 < x_3 = x_4 = \xi_2 < \xi_3 < \xi_4 < x_5 = \cdots = x_8 = \xi_5$. The multiplicity 4 is not indexed with $a$ because it is $> k$, *i.e.* it automatically implies $x_5 = \cdots = x_8 = \xi_5$.

***Definition 5***. — Given a hyperbolic polynomial $P$ call *roots of class B* (resp. *roots of class A*) its roots of multiplicity $< k$ which coincide with roots of $P^{(k)}$ (resp. all its other roots). In a CV the roots of class B correspond to multiplicities indexed by $a$.

***Definition 6***. — For a given CV $\vec{v}$ call *stratum* of $\Pi$ (defined by $\vec{v}$) its subset of polynomials $P$ with configuration of the roots of $P$ and $P^{(k)}$ defined by $\vec{v}$.

The aim of the present paper is to prove the following

***Theorem 7***. — *All strata of this stratification are smooth contractible real semi-algebraic sets. Their closures are real algebraic varieties.*

The theorem is proved in Section 5. That the strata mentioned above define a true stratification is shown in Remark 15.

***Remark 8***. — It is shown in [**KoSh**], Theorem 4.4, that every a priori admissible CV defines a non-empty connected stratum. The essentially new result of the present paper is the proof not only of connectedness but of contractibility. In [**Ko5**] the notion of a priori admissible CV is generalized in the case of not necessarily hyperbolic polynomials and it is shown there that all such CVs are realizable by the arrangements of the real roots of polynomials $P$ and of their derivatives $P^{(k)}$ (the position and multiplicity of the complex roots is not taken into account there).

***Notation 9***. — We denote by $D(i,j)$ the *discriminant set* $\{a \in \mathbf{R}^n \mid \operatorname{Res}(P^{(i)}, P^{(j)})=0\}$ (recall that for $a \in \Pi$ one has $\operatorname{Res}(P^{(i)}, P^{(j)}) = 0$ if and only if $P^{(i)}$ and $P^{(j)}$ have a common root).

Let $a_0 \in D(0,k) \cap \operatorname{Int}\Pi$ be such that for $a_0$ there hold exactly $s$ equalities of the form $x_j^{(k)} = x_i$, with $s$ different indices $j$ and $s$ different indices $i$.

***Proposition 10***. — *In a neighbourhood of the point $a_0$ the set $D(0,k)$ is locally the union of $s$ smooth hypersurfaces intersecting transversally at $a_0$.*

All propositions are proved in Section 4. The proposition can be generalized in the following way. Suppose that at a point $a_0$ lying in the interior of $\Pi$ there hold exactly $s$ equalities $x_j^{(k_i)} = x_i$, with $s$ different indices $i$ and $s$ different couples $(k_i, j)$.

***Proposition 11***. — *In a neighbourhood of the point $a_0$ these $s$ equalities define $s$ smooth hypersurfaces intersecting transversally at $a_0$.*

***Remark 12***. — It is shown in [**Ko3**] that for each $q$-tuple of positive integers $m_j$ with sum $n$ the subset $T$ of $\Pi$ (we call it a *stratum* of $\Pi$ defined by the *multiplicity vector* $(m_1, \ldots, m_q)$, not by a CV) consisting of polynomials with distinct roots $y_i$, of multiplicities $m_i$, is a smooth variety of dimension $q$ in $\mathbf{R}^n$.

Denote by $T$ a stratum of $\Pi$ defined by a multiplicity vector. Fix a point $G \in T$. Suppose that at $G$ there are $s$ among the roots $y_j$ which are of class B. Suppose that one has $m_i < k$ for all $i$. The condition $m_i < k$ implies that all points from $D(0,k) \cap T$ close to $G$ result from roots of $P^{(k)}$ coinciding with roots of $P$ of class B.

***Proposition 13***. — *In a neighbourhood of the point $G$ the set $D(0,k) \cap T$ is locally the union of $s$ smooth codimension 1 subvarieties of $T$ intersecting transversally at $G$.*

***Remarks 14***. — 1) A stratum of $\Pi$ of codimension $\kappa \leqslant k$ defined by $\kappa$ equalities of the form $x_i = \xi_j$ (*i.e.* $P$ has no multiple root) has a tangent space transversal to the space $Oa_{n-\kappa+1} \ldots a_n$. Indeed, the roots $\xi_j$ depend smoothly on $a_1, \ldots, a_{n-k}$, and the conditions $P(\xi_j, a) = 0$ allow one to express $a_{n-\kappa+1}, \ldots, a_n$ as smooth functions of $a_1, \ldots, a_{n-\kappa}$ (use Vandermonde's determinant with distinct arguments $\xi_1, \ldots, \xi_\kappa$). It would be nice to prove or disprove the statements:

A) this property holds without the assumption $\kappa \leqslant k$ and that $P$ has no multiple root;

B) the limit of the tangent space to a stratum, when a stratum in its closure is approached, exists and is transverse to the space $Oa_{n-\kappa+1} \ldots a_n$.

For $n = 4$ and $n = 5$ this seems to be true, see [**Ko1**] and [**Ko2**]. The statements would be a generalization of such a transversality property of the strata of $\Pi$ defined by multiplicity vectors, not by CVs (proved in [**Ko3**], Theorem 1.8; see Remark 12). Outside $\Pi$ the first statement is not true – for $n = 4$, $a_1 = 0$, the discriminant set $D(0,2)$ has a Whitney umbrella singularity at the origin and there are points where its tangent space is parallel to $Oa_4$; this can be deduced from [**Ko1**] (see Section 3 and Lemma 29 in it).

2) In [**KoSh**], [**Ko1**] and [**Ko2**] a stratification of $\Pi$ defined by the arrangement of all roots of $P, P', \ldots, P^{(n-1)}$ is considered (the initial idea to consider this stratification belongs to B.Z. Shapiro). The results of the present paper cannot be transferred directly to that case for two reasons:

a) for $n \geqslant 4$ not all arrangements consistent with (1) are realized by hyperbolic polynomials and it is not clear how to determine for any $n \in \mathbf{N}^*$ the realizable ones (*e.g.* for $n = 4$ only 10 out of 12 such arrangements are realized, see [**KoSh**] or [**Ko1**]; for $n = 5$ only 116 out of 286, see [**Ko1**]); the reason for this is clear – a monic polynomial has only $n$ coefficients that can be varied whereas there are $n(n+1)/2$ roots of $P, P', \ldots, P^{(n-1)}$;

b) for $n \geqslant 4$ there are *overdetermined strata*, *i.e.* strata on which the number of equalities between any two of the roots of $P, P', \ldots, P^{(n-1)}$ is greater than the codimension of the stratum.

In Section 3 we prove two technical lemmas (and their corollaries) used in the proof of the theorem and the propositions. Section 2 is devoted to the dimension of a stratum and its relationship with the CV defining it. The above propositions are just the first steps in the study of the set $D(0,1) \cup D(0,k)$ (and, more generally, of the set $D(0,1) \cup \cdots \cup D(0,n-1)$) at a point of $\Pi$.

## 2. Configuration vectors and dimensions of strata

In this section we recall briefly results some of which are from [**KoSh**]:

1) Call *excess of multiplicity* of a CV the sum $m = \sum(m_j - 1)$ taken over all multiplicities $m_j$ of distinct roots of $P$. A *stratum* of codimension $i$ is defined by a CV which has exactly $i - m$ letters $a$ as indices, *i.e.* the polynomial $P$ has exactly $i - m$ distinct roots of class B.

2) A stratum of codimension $i$ is locally a smooth real algebraic variety of dimension $n - i$ in $\mathbf{R}^n$.

3) In what follows we say a stratum of codimension $i$ to be of dimension $n-i-2$. We decrease its dimension in $\mathbf{R}^n$ by 2 to factor out the possible shifting of the variable $x$ by a constant and the one-parameter group of transformations $x \mapsto \exp(t)x$, $a_j \mapsto \exp(jt)a_j$, $t \in \mathbf{R}$; both of them leave CVs unchanged. This allows one to consider the

family $P$ only for $a_1 = 0$, $a_2 = -1$ (if $a_1 = 0$, then there are no hyperbolic polynomials for $a_2 > 0$ and for $a_2 = 0$ the only one is $x^n$).

4) In accordance with the convention from 3), it can be deduced from 1) that the CVs defining strata of dimension $\delta$ are exactly the ones in which the polynomial $P$ has $\delta + 2$ distinct roots of class A, *i.e.* these are CVs having $\delta + 2$ components which are multiplicities of roots of $P$ not indexed by the letter $a$.

5) A point of a stratum of codimension $i > 1$ defined by a CV $\vec{v}$ belongs to the closure of any stratum of codimension $i - 1$ whose CV $\vec{w}$ is obtained from $\vec{v}$ by means of one of the following three operations:

i) if $\vec{v} = (A, l_a, B)$, $l \leqslant k - 1$, $A$ and $B$ are non-void, then $\vec{w} = (A, l, a, B)$ or $\vec{w} = (A, a, l, B)$;

ii) if $\vec{v} = (A, r_a, B)$, $r \leqslant k - 1$, $A$ and $B$ are non-void, then $\vec{w} = (A, r', r''_a, B)$ or $\vec{w} = (A, r'_a, r'', B)$, $r' > 0$, $r'' > 0$, $r' + r'' = r$.

iii) if $\vec{v} = (A, r, B)$, then $\vec{w} = (A, C, B)$ where $C$ is a CV defining a stratum of dimension 0 in $\mathbf{R}^r$, see 4).

6) It follows from the definition of the codimension of a stratum that the three possibilities *i)*, *ii)* and *iii)* from 5) are the only ones to increase by 1 the dimension of a stratum $S$ when passing to a stratum containing $S$ in its closure. Indeed, one has to increase by 1 the number of roots of class A, see 4). If to this end one has to change the number or the multiplicities of the roots of class B, then there are no possibilities other than *i)* and *ii)*. If not, then exactly one root $x_i$ of class A must bifurcate, the roots stemming from it and the roots of $P^{(k)}$ close to $x_i$ must define an a priori admissible CV (they must satisfy conditions (1)), and among these roots there must be exactly two of class A. Hence, the bifurcating roots must define a CV of dimension 0 in $\mathbf{R}^r$, see 4).

***Remark 15***. — The strata define a true stratification in the sense that they are connected components of differences of closed sets of a filtration. Indeed, the filtration is defined by the codimension of the strata. Contractibility (hence, connectedness) follows from Theorem 7. To obtain a stratum as a difference of closed sets one can represent it as the difference between its closure $Z$ and the closure of the union of all strata of lower dimension belonging to $Z$.

## 3. Two technical lemmas and their corollaries

For a monic strictly hyperbolic polynomial $P$ of degree $n$ consider the roots $x_j^{(k)}$ of $P^{(k)}$ as functions of the roots $x_i$ of $P$. Hence, these functions are smooth because the roots $x_j^{(k)}$ are simple, see Remark 3.

***Lemma 16***. — *For $i = 1, \ldots, n$, $k = 1, \ldots, n-1$, $j = 1, \ldots, n-k$ one has $\partial(x_j^{(k)})/\partial(x_i) > 0$.*