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AN EXPLICIT CYCLE REPRESENTING THE FULTON-JOHNSON CLASS, I

by

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Abstract. — For a singular hypersurface X in a complex manifold we prove, under certain conditions, an explicit formula for the Fulton-Johnson classes in terms of obstruction theory. In this setting, our formula is similar to the expression for the Schwartz-MacPherson classes provided by Brasselet and Schwartz. We use, on the one hand, a generalization of the virtual (or GSV) index of a vector field to the case when the ambient space has non-isolated singularities, and on the other hand a Proportionality Theorem for this index, similar to the one due to Brasselet and Schwartz.

***Résumé* (Une description explicite de la classe de Fulton Johnson, I).** — Pour une hypersurface singulière X d'une variété complexe, et dans certaines conditions, nous montrons une formule explicite pour les classes de Fulton-Johnson en termes de théorie d'obstruction. Dans ce contexte notre formule est similaire à l'expression des classes de Schwartz-MacPherson donnée par Brasselet et Schwartz. Nous utilisons, d'une part, une généralisation de l'indice virtuel (ou GSV-indice) d'un champs de vecteurs au cas où l'espace ambiant a des singularités non-isolées et, d'autre part, un Théorème de Proportionnalité pour cet indice, similaire à celui dû à Brasselet et Schwartz.

1. Introduction

There are several different ways to generalize the Chern classes of complex manifolds to the case of singular varieties. Among them are the Schwartz-MacPherson classes [**5, 16, 20**] and the Fulton-Johnson classes [**8, 9**]. Each one of them is defined in a relevant context and has its own interest and advantages. The construction in [**5, 20**] provides a geometric interpretation of the Schwartz-MacPherson classes via

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obstruction theory. This approach is very useful for understanding what these classes measure.

The motivation for this work is to give such a geometric interpretation of the Fulton-Johnson classes, in the spirit of [**5, 20**]. Here we prove that if $X \subset M$ is a singular complex analytic hypersurface of dimension n, defined by a holomorphic function on a manifold M, then the Fulton-Johnson classes can be regarded as "weighted" Schwartz-MacPherson classes.

In order to explain our result more precisely, let us consider a complex analytic manifold M of dimension m, and a compact singular analytic subvariety $X \subset M$. Let us endow M with a Whitney stratification adapted to X [**24**], and consider a triangulation (K) of M compatible with the stratification. We denote by (D) a cellular decomposition of M dual to (K). Let us notice that if the $2q$-cell d_α of (D) meets X, it is dual of a $2(m-q)$-simplex σ_α of (K) in X.

We recall that in her definition of Chern classes, M.H. Schwartz considers particular stratified r-frames v^r tangent to M, called radial frames. They have no singularity on the $(2q-1)$-skeleton of (D), with $q = m - r + 1$, and isolated singularities on the $2q$-cells d_α, at their barycenter $\{\widehat{\sigma}_\alpha\} = d_\alpha \cap \sigma_\alpha$. Let us denote by $I(v^r, \widehat{\sigma}_\alpha)$ the index of the r-frame v^r at $\widehat{\sigma}_\alpha$.

The result of [**5**] tells us that the Schwartz-MacPherson class $c_{r-1}(X)$ of X of degree $(r-1)$ is represented in $H_{2(r-1)}(X)$ by the cycle

$$\sum_{\substack{\sigma_\alpha \subset X, \\ \dim \sigma_\alpha = 2(r-1)}} I(v^r, \widehat{\sigma}_\alpha) \cdot \sigma_\alpha$$

In this article we prove:

Theorem 1.1. — *Let us assume that $X \subset M$ is a hypersurface, defined by $X = f^{-1}(0)$, where $f : M \to \mathbb{D}$ is a holomorphic function into an open disc around 0 in $\mathbb{C}$. For each point $a \in X$ let F_a denote a local Milnor fiber, and let $\chi(F_a)$ be its Euler-Poincaré characteristic. Then the Fulton-Johnson class $c^{FJ}_{r-1}(X)$ of X of degree $(r-1)$ is represented in $H_{2(r-1)}(X)$ by the cycle*

$$\sum_{\substack{\sigma_\alpha \subset X, \\ \dim \sigma_\alpha = 2(r-1)}} \chi(F_{\widehat{\sigma}_\alpha}) I(v^r, \widehat{\sigma}_\alpha) \cdot \sigma_\alpha \tag{1.1}$$

On the other hand, the question of understanding the difference between the Schwartz-MacPherson and the Fulton-Johnson classes has been addressed by several authors, and this led to the concept of *Milnor classes*, defined by $\mu_*(X) = (-1)^{n+1}\big(c_*(X) - c_*^{FJ}(X)\big)$, $n = \dim X$, see for instance [**1, 3, 19, 25**]. Let us define the local Milnor number of X at the point $a \in X$ by $\mu(X, a) = (-1)^{n+1}(1 - \chi(F_a))$; it coincides with the usual Milnor number of [**17**] when a is an isolated singularity of X. It is non zero only on the singular set Σ of X. We have the following immediate consequence of Theorem 1.1:

Corollary 1.2. — *Under the assumptions of Theorem 1.1, the Milnor class $\mu_{r-1}(X)$ in $H_{2(r-1)}(X)$ is represented by the cycle*

$$\sum_{\substack{\sigma_\alpha \subset \Sigma \\ \dim \sigma_\alpha = 2(r-1)}} \mu(X, \widehat{\sigma}_\alpha)\, I(v^r, \widehat{\sigma}_\alpha) \cdot \sigma_\alpha \tag{1.2}$$

One of the key ingredients we use for proving the Theorem 1.1 is a *Proportionality Theorem* for the index of vector fields and frames on singular varieties, similar to the one given in [**5**]. In order to establish it we were led to defining *the local virtual index* at an isolated zero of a smooth vector field on a complex hypersurface with (possibly) non-isolated singularities. This is a generalization of the indices defined previously in [**4, 12, 15**]. We call it "local" virtual index to distinguish it from the "global" virtual index at a whole component of the singular set, as studied in [**4**]

We notice that for hypersurfaces with isolated singularities one also has the homological index of [**11**], which coincides with the index in [**12**]. It would be interesting to know whether our generalized virtual (or GSV) index coincides with the generalized homological index in [**10**] when the ambient space has non-isolated singularities.

Our formulae can also be obtained in another way, using the MacPherson morphism c_* (see [**16**]) together with the Verdier specialization map of constructible functions [**23**], since one knows (see for instance [**19**]) that the Fulton-Johnson and the Milnor classes are image by the morphism c_* of certain constructible functions. The advantage of our construction here is to provide a geometric and explicit point of view, which can be used to study the general case. This is being done in [**6**].

2. The local virtual index of a vector field

Let $(X, 0)$ be a hypersurface germ in an open set $\mathcal{U} \subset \mathbb{C}^{n+1}$, defined by a holomorphic function $f : (\mathcal{U}, 0) \to (\mathbb{C}, 0)$. Let us endow $\mathcal{U}$ with a Whitney stratification $\{V_i\}$ compatible with X and let us consider the subspace E of the tangent bundle $T\mathcal{U}$ of $\mathcal{U}$ consisting of the union of the tangent bundles of all the strata.:

$$E = \bigcup_{V_i} TV_i \tag{2.3}$$

A section of $T\mathcal{U}$ whose image is in E is called a *stratified vector field* on $\mathcal{U}$.

Let v be a stratified vector field on $(X, 0)$ with an isolated singularity (zero) at $0 \in X$. We want to define an index of v at $0 \in X$ which coincides with the *GSV*-index of [**12**] (or the virtual index in [**4**]) when 0 is also an isolated singularity of X. For this, let us consider a (sufficiently small) ball B_ε around $0 \in \mathcal{U}$ and denote by $\mathcal{T}$ the Milnor tube $f^{-1}(D_\delta) \cap B_\varepsilon$, where D_δ is a (sufficiently small) disc around $0 \in \mathbb{C}$. We let $\partial\mathcal{T}$ be the "boundary" $f^{-1}(C_\delta) \cap B_\varepsilon$ of $\mathcal{T}$, $C_\delta = \partial D_\delta$.

Let r be the radial vector field in $\mathbb{C}$ whose solutions are straight lines converging to 0. It can be lifted to a vector field $\widetilde{r}$ in $\mathcal{T}$, whose solutions are arcs that start in

$\partial\mathcal{T}$ and finish in X; since the corresponding trajectories in $\mathbb{C}$ are transversal to all the circles (C_η) around $0 \in \mathbb{C}$ of radius $\eta \in]0, \delta[$, it follows that the solutions of $\widetilde{r}$ are transversal to all the tubes $f^{-1}(C_\eta)$. This vector field $\widetilde{r}$ defines a C^∞ retraction ξ of $\mathcal{T}$ into X, with X as fixed point set. The restriction of ξ to any fixed Milnor fibre $F = f^{-1}(t_0) \cap B_\varepsilon$, $t_0 \in C_\delta$, provides a continuous map $\pi : F \to X$, which is surjective and it is C^∞ over the regular part of X. We call such map ξ, or also π, a *degenerating map* for X (this was called a "collapsing map" in [**14**]). Since the singular set Σ of X is a Zariski closed subset of X, we notice that we can choose the lifting $\widetilde{r}$ so that $\pi^{-1}(X_{\rm reg})$ is an open dense subset of F, where $X_{\rm reg}$ is the regular part $X_{\rm reg} = X \setminus \Sigma$.

We want to use π to lift the stratified vector field v on X to a vector field on F. Firstly, let us consider the case where X has an isolated singularity at 0. The map π is a diffeomorphism restricted to a neighbourhood $N \subset F$ of $F \cap \partial B_\varepsilon$. Then v can be lifted to a non-singular vector field on N and extended to the interior of F with finitely many singularities, by elementary obstruction theory. By definition [**12**], the total Poincaré-Hopf index of this vector field on F is the GSV-index of v on X.

We want to generalize this construction to the case when the singularity of X at 0 is not necessarily isolated. Let us consider $(X, 0)$ as above, a possibly non-isolated germ. We fix a Milnor fibre $F = f^{-1}(t_o) \cap B_\varepsilon$ for some $t_o \in C_\delta$. Given a point $x \in F$, we let γ_x be the solution of $\widetilde{r}$ that starts at x. The end-point of γ_x is the point $\pi(x) \in X$. We parametrize this arc γ_x by the interval $[0, 1]$, with $\gamma_x(0) = x$ and $\gamma_x(1) = \pi(x)$. We assume that this interval $[0, 1]$ is the straight arc in $\mathbb{C}$ going from t_o to 0, so that for each $t \in [0, 1[$, the point $\gamma_x(t)$ is in a unique Milnor fibre $F_t = f^{-1}(t) \cap B_\varepsilon$. The family of tangent spaces to F_t at the points $\gamma_x(t)$ defines a 1-parameter family of n-dimensional subspaces of $\mathbb{C}^{n+1}$, $\{TF_t\}_{\gamma_x(t)}$. By [**18**] we may assume that the Whitney stratification $\{V_i\}$ satisfies the strict Thom w_f-condition. This implies that for each trajectory $\gamma_x(t)$ the corresponding family $\{TF_t\}_{\gamma_x(t)}$ has a well defined limit space $\Lambda_{\pi(x)}$, *i.e.* it converges to an n-plane $\Lambda_{\pi(x)} \subset T_{\pi(x)}(\mathcal{U})$ when $t \to 1$. Hence one has an identification $T_xF \cong \Lambda_{\pi(x)}$ which defines an isomorphism of vector spaces. Moreover, since w_f implies the Thom a_f-condition one has that the limit space $\Lambda_{\pi(x)}$ contains the space $T_{\pi(x)}V_i$ tangent to the stratum that contains $\pi(x)$. Therefore the vector $v(\pi(x))$ can be lifted to a vector $\widetilde{v}(x) \in T_xF$. This vector field $\widetilde{v}$ is non-singular over the inverse image of $X_{\rm reg}$, which is open and dense in F. Also $\widetilde{v}$ is non-zero on a neighbourhood of $F \cap \partial B_\varepsilon$, since v is assumed to have an isolated singularity at 0. Furthermore, by the w_f-condition the vector field $\widetilde{v}$ is continuous, so it has a well defined Poincaré-Hopf index in F. The w_f-condition also implies that the angle between $v(\pi(x))$ and $\widetilde{v}(x)$ is small. That is, given any $\alpha > 0$ small, we can choose δ sufficiently small with respect to α so that the angle between $v(\pi(x))$ and $\widetilde{v}(x)$ is less than α. This implies that if we replace $\widetilde{v}$ by some other lifting of v, the induced vector fields on F are homotopic. Since f induces a locally trivial fibration over the punctured disc $D_\delta \setminus 0$, then the homotopy class of $\widetilde{v}$ does not depend on the choice of the Milnor fibre. So we obtain:

Proposition 2.1. — *The Poincaré-Hopf index of $\widetilde{v}$ in F depends only on $X \subset \mathcal{U}$ and the vector field v. It is independent of the choices of the Milnor fibre F as well as the liftings involved in its definition. We call this integer the local virtual index of v on X at 0, and we denote it by $\mathcal{I}_{\mathrm{v}}(v, 0, X)$.*

In other words, the index $\mathcal{I}_{\mathrm{v}}(v, 0, X)$ is the obstruction $\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon))$ to the extension of the lifting $\widetilde{v}$ as a section of TF without singularity on $\pi^{-1}(B_\varepsilon(0))$.

Let us consider now the case where w is a stratified vector field transversal to the boundary $S_\varepsilon = \partial(B_\varepsilon)$ of every small ball B_ε, *pointing outwards*; it has a unique singular point (inside B_ε) at 0. The Poincaré-Hopf index of w at the point 0, denoted by $I(w, 0)$, is equal to 1, computed either in M or in the stratum $V_i(0)$ of X containing 0 (if the dimension of $V_i(0)$ is more than 0). The lifting $\widetilde{w}$ is a section of TF on $\pi^{-1}(S_\varepsilon) = F \cap S_\varepsilon$, pointing outwards $\pi^{-1}(B_\varepsilon) = F \cap B_\varepsilon$.

Let us denote by T^*F the fiber bundle over F which is TF minus the zero section. The obstruction to the extension of $\widetilde{w}$ as a section of T^*F inside $\pi^{-1}(B_\varepsilon)$ is equal to the Euler-Poincaré characteristic of the Milnor fiber, *i.e.*

$$\mathrm{Obs}(\widetilde{w}, T^*F, \pi^{-1}(B_\varepsilon)) = \chi(F). \tag{2.4}$$

We obtain:

Proposition 2.2. — *If w is a stratified vector field pointing outwards the ball B_ε along its boundary $S_\varepsilon = \partial(B_\varepsilon)$, then its local virtual index equals the Euler-Poincaré characteristic of the Milnor fiber:*

$$\mathcal{I}_{\mathrm{v}}(w, 0, X) = \chi(F) = 1 + (-1)^n \mu(X, 0).$$

In the sequel, for any vector bundle ξ over a space B, we will denote by ξ^* the bundle over B which is ξ minus its zero section.

3. Proportionality Theorems

Let us consider again a stratified vector field v defined on the ball $B_\varepsilon \subset \mathcal{U}$, with a unique singularity at 0. We assume further that v is constructed by the radial extension process of M. H. Schwartz [**20**]. This means, essentially, that if V_j is any stratum containing $V_i(0)$ in its closure, then the vector field v is transversal to the boundary of every tubular neighbourhood of $V_i(0)$ in X, pointing outwards. The Poincaré-Hopf index of v, computed in $V_i(0)$ and denoted $I(v, 0)$, can be any integer, and the fact that v is constructed by radial extension implies that $I(v, 0)$ equals the Poincaré-Hopf index of v computed in $\mathcal{U}$. We shall call v a *vector field constructed by radial extension*, or simply *a radial vector field* if this does not lead to confusion, as in Theorem 3.1 below. If the stratum $V_i(0)$ has dimension 0, this implies that v is