

ADE SURFACE SINGULARITIES, CHAMBERS AND TORIC VARIETIES

by

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Abstract. — We study the link between the positive divisors supported on the exceptional divisor of the minimal resolution of a rational double point and the root systems of Dynkin diagrams. Then, we calculate the toric variety corresponding to the fundamental Weyl chamber.

Résumé (Singularités ADE des surfaces, chambres et variétés toriques). — Nous étudions le lien entre les diviseurs positifs à support sur le diviseur exceptionnel de la résolution minimale d'un point double rationnel et les systèmes de racine des diagrammes de Dynkin. Puis, nous calculons la variété torique correspondant à la chambre fondamentale de Weyl.

1. Introduction

A singularity of a normal analytic surface is rational if the geometric genus of the surface doesn't change by a resolution of the singularity. These singularities are rather simple among surface singularities since they are absolutely isolated and their resolutions have some nice combinatoric properties. A classification of rational singularities is done by the dual graph of the minimal resolution according to their multiplicities (see [11] for details and related references).

First, DuVal observed that the dual graph of the minimal resolution of a rational singularity of multiplicity 2, called rational double point, with algebraically closed field is one of the Dynkin diagrams A_n , D_n , E_6 , E_7 and E_8 , briefly ADE diagrams (see [2] or [4]). This means that the intersection matrix associated to the dual graph of the minimal resolution of a rational double point is the same as the Cartan matrix of the corresponding Dynkin diagram.

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The negative definiteness of the intersection matrix of the exceptional divisor of a resolution of a normal surface singularity permits us to study on a set of certain positive divisors supported on the exceptional divisor, which will be called the semigroup of Lipman. By using this set, we can associate a toric variety with a weighted graph whose intersection matrix is negative definite (see [1]).

In this work, motivated by a question appeared in [9], we give a geometric construction of the roots of an ADE diagram, listed in [3] (see Planche I,IV,V,VI,VII). Following [14], we observe that the semigroup of Lipman associated with an ADE diagram is the same as the fundamental Weyl chamber of the corresponding root system. In the last section, using [1], we describe the toric variety corresponding to the fundamental Weyl chamber of an ADE diagram (see [1], [15]).

2. Rational Singularities

Let S be a germ at ξ of a complex two dimensional normal space with a singularity at ξ . A *resolution* of S is a complex nonsingular surface with a proper map $\pi : X \rightarrow S$ such that its restriction to $X - \pi^{-1}(\xi)$ is an isomorphism and $X - \pi^{-1}(\xi)$ is dense in X . A resolution $\pi : X \rightarrow S$ is called *minimal resolution* if any other resolution $\pi' : X' \rightarrow S$ factorizes by π . It is well known that the exceptional divisor $E = \pi^{-1}(\xi)$ of π is connected and of dimension 1 (see [7], theorem V.5.2). Let us denote by $E_1, \dots, E_n$ the irreducible components of E . The *intersection matrix* $M(E)$ associated with E is defined by the intersection $(E_i \cdot E_j)$ of the components E_i and E_j , which is the intersection number of E_i and E_j if $i \neq j$, and the first Chern class of the normal bundle to E_i if $i = j$. It is a negative definite matrix (see [13]).

Let $\mathcal{G}$ denote the free abelian group generated by the irreducible components of E :

$$\mathcal{G} = \left\{ \sum_{i=1}^n m_i E_i, m_i \in \mathbb{Z} \right\}.$$

The elements of $\mathcal{G}$ are called the *divisors supported on E* . The support of a divisor $Y = \sum_i m_i E_i$ is the set of the components for which $m_i \neq 0$. In the free abelian group $\mathcal{G}$, the intersection matrix $M(E)$ defines a symmetric bilinear form. We shall denote $(Y \cdot Z)$ the value of this bilinear form on a pair (Y, Z) of elements in $\mathcal{G}$. An element of $\mathcal{G}$ in which all the coefficients are non-negative and at least one is positive, is called a *positive divisor*.

Theorem 2.1 (see [2]). — *The singularity ξ of S is a rational singularity if and only if the arithmetic genus $\frac{1}{2}(Y \cdot Y + \sum_{i=1}^n m_i(w_i - 2)) + 1$ of each positive divisor $Y = \sum_{i=1}^n m_i E_i$ in $\mathcal{G}$ is ≤ 0 where $w_i = -(E_i \cdot E_i)$.*

Assume that $\pi : X \rightarrow S$ is a resolution of a normal surface singularity which is not necessarily rational. Let f be an element of the maximal ideal $\mathcal{M}$ of $\mathcal{O}_{S,\xi}$. Then the divisor $(\pi^* f)$ of f on X is written as $(\pi^* f) = Y + T_f$ where Y is a positive divisor supported on the exceptional divisor E of π and T_f , called the strict transform of f

by π , intersects E in finitely many point at most. Since $((\pi^*f) \cdot E_i) = 0$ for all i , we obtain $(Y \cdot E_i) \leq 0$ for all i . The inverse is true when the singularity ξ is rational. We mean that, if Y is a positive divisor on X such that $(Y \cdot E_i) = -(T \cdot E_i)$ for all i , then there exists a function f in $\mathcal{M}$ such that $(\pi^*f) = Y + T$ (see [2]). Now, as in [12] (see section 18), let us consider the set

$$\mathcal{E}^+(E) = \{Y \in \mathcal{G} \mid (Y \cdot E_i) \leq 0 \text{ for all } i\}$$

By [18], this set is not empty. It is an additive semigroup: For $Y_1, Y_2 \in \mathcal{E}^+(E)$, we have $Y_1 + Y_2 \in \mathcal{E}^+(E)$.

Definition 2.2. — The set $\mathcal{E}^+(E)$ is called the semigroup of Lipman.

Since E is connected, for all $Y = \sum m_i E_i$ in $\mathcal{E}^+(E)$, we have $m_i \geq 1$ for all i . A partial order on $\mathcal{E}^+(E)$ is defined as follows: For two elements $Y_1 = \sum_{i=1}^n a_i E_i$ and $Y_2 = \sum_{i=1}^n b_i E_i$ of $\mathcal{E}^+(E)$, we say $Y_1 \leq Y_2$ if $a_i \leq b_i$ for all i . The smallest element of this set is called *the fundamental cycle* of the resolution π . The proposition 4.1 in [10], gives the following algorithm to construct the fundamental cycle of a given E :

Let us denote by Z the fundamental cycle of π . Consider $Z_1 = \sum_{i=1}^n E_i$. If $(Z_1 \cdot E_i) \leq 0$ for all i , then $Z_1 = Z$. If else, there exists an E_{i_1} such that $(Z_1 \cdot E_{i_1}) > 0$; in this case, we put $Z_2 = Z_1 + E_{i_1}$ and we see whether $(Z_2 \cdot E_i) \leq 0$ for all i . The term Z_j , ($j \geq 1$), of the sequence satisfies, either $(Z_j \cdot E_i) \leq 0$ for all i , then we put $Z = Z_j$, or there is an irreducible component E_{i_j} such that $(Z_j \cdot E_{i_j}) > 0$, then we put $Z_{j+1} = Z_j + E_{i_j}$. Thus the fundamental cycle of π is the first cycle Z_k of this sequence such that $(Z_k \cdot E_i) \leq 0$ for all i . By the same method, we can construct all other elements of $\mathcal{E}^+(E)$ (see [14] or [17]).

The following result of Artin characterize what an exceptional divisor of a resolution of a rational singularity looks like:

Theorem 2.3 (see [2]). — *A singularity of a normal analytic surface in $\mathbb{C}^N$ is rational if and only if the arithmetic genus of the fundamental cycle of the exceptional divisor of a resolution of the singularity vanishes.*

This gives:

Corollary 2.4 (see [2]). — *The exceptional divisor of any resolution of a rational singularity is normal crossing, with each E_i nonsingular and of genus zero, and any two distinct components intersect transversally at most in one point.*

A proof of this corollary can be found also in [17].

Then the *dual graph* associated with the exceptional divisor of a resolution of a rational singularity, in which each E_i is represented by a vertex and each intersection point is represented by an edge between the vertices corresponding to the intersecting components, is a tree. Each vertex in the dual graph is weighted by $-(E_i \cdot E_i)$. Conversely, with a given weighted graph, by plumbing, we can associate a *configuration*

of curves embedded in a nonsingular surface and, if such a configuration of curves satisfies theorem 2.3, its contraction gives a rational singularity of a normal analytic surface (see [6], [11]).

Example 2.5. — A configuration of curves associated with an ADE diagram is contracted to a rational singularity of a normal analytic surface.

Moreover, we have:

Proposition 2.6 (see [2]). — *Let $\pi : X \rightarrow S$ be the minimal resolution of the rational singularity ξ of S . Then the multiplicity of S at ξ equals $-(Z \cdot Z)$ where Z is the fundamental cycle of π .*

Recall that the minimal resolution is characterized by $(E_i \cdot E_i) \leq -2$ for all irreducible components E_i of the exceptional divisor. A rational double point is a rational singularity for which the fundamental cycle of the minimal resolution satisfies $(Z \cdot Z) = -2$. We know that a rational double point of a surface is defined by the power series with the form $f(x, y) + z^2 = 0$. By using the results given above, we deduce:

Proposition 2.7 (see [2] or [4]). — *A normal analytic surface singularity is a rational double point if and only if the exceptional divisor of the minimal resolution of the singularity is a configuration of curves associated with one of the ADE diagrams.*

3. Root systems of rational double points

There is a well known construction of ADE diagrams starting from a semisimple Lie algebra. In this section, we are interested in the inverse of that construction, as suggested in [9]. We will see that, using the geometry of a Dynkin diagram, we can obtain the roots of the corresponding semisimple Lie algebra. This gives a partial answer to the question of Ito and Nakamura (see [9], p.194).

Let V be an euclidean space endowed with a positive definite symmetric bilinear form $(,)$. A reflection s on V is an orthogonal transformation $s : V \rightarrow V$ such that, for $v \in V$, $s(v) = -v$ and it fixes pointwise the hyperplane $H_v = \{u \in V \mid (u, v) = 0\}$ of V . We can describe the reflection by the formula $s_v(u) = u - \frac{2(u, v)}{(v, v)}v$.

Definition 3.1. — A subset R of V is called a root system if

- (i) it is finite, generates V and doesn't contain 0,
- (ii) for every $v \in R$, there exists a unique reflection s_v such that $s_v(R) = R$,
- (iii) for every $v \in R$, the only multiples of v in R are $\pm v$,
- (iv) for $u, v \in R$, we have $\frac{2(u, v)}{(v, v)} \in \mathbb{Z}$.

The finite group generated by the reflections is called the Weyl group. See [8] for more details.

In what follows, E will denote a configuration of curves associated with an ADE diagram, called ADE configuration. Now, following [14], (see p.158), we want to establish the relation between the root systems and the semigroup of Lipman of E . Denote by $E_1, \dots, E_n$ the irreducible components of E . We know that $(E_i \cdot E_j)$ equals -2 if $i = j$ and equals 0 or 1 if $i \neq j$ (see [4] or [12]). Now, consider the following subset of $\mathcal{G}$:

$$R(E) = \{Y \in \mathcal{G} \mid (Y \cdot Y) = -2\}.$$

Proposition 3.2 (see [14]). — *The set $R(E)$ is a root system.*

Replacing the inner product in the definition above by the symmetric bilinear form defined by the intersection matrix $M(E)$, we can see that $R(E)$ satisfies the conditions of the definition above.

We will call *root divisors* the elements of $R(E)$. By definition, $E_1, \dots, E_n$ and $-E_1, \dots, -E_n$ are root divisors but $E_i - E_j$ is not a root divisor since $(E_i - E_j \cdot E_i - E_j) \neq -2$ for any $i \neq j$. Let us denote $B = \{E_1, \dots, E_n\}$. We can see that B is a vector space basis of $R(E)$ in $\mathcal{G} \otimes_{\mathbb{Z}} \mathbb{R}$ and every element Y in $R(E)$ can be written as the sum of E_i 's with coefficients all nonnegative or all nonpositive (compare with [8], pp.47-48). If we denote by $R^+(E)$ the set of the elements of $R(E)$ with coefficients all nonnegative, then we have $R(E) = R^+(E) \cup (-R^+(E))$.

Proposition 3.3 (see [14]). — *Let $Z = \sum_{i=1}^n a_i E_i$ be the fundamental cycle of E . Then, for each root divisor $Y = \sum_{i=1}^n m_i E_i$ in $R(E)$, we have $m_1 \leq a_1, \dots, m_n \leq a_n$.*

The fundamental cycle is called the highest (or biggest) root divisor in $R(E)$.

Proof. — Since E is the exceptional divisor of the minimal resolution of a rational double point, we have $(Z \cdot Z) = -2$. So $Z \in R(E)$. Assume that there is a positive divisor Y in $R(E)$ such that $Y > Z$ and $(Y \cdot Y) = -2$. So we have $Y = Z + D$ where D is a positive divisor. This gives $(Y \cdot Y) = (Z \cdot Z) + 2(Z \cdot D) + (D \cdot D)$. Thus $2(Z \cdot D) = -(D \cdot D)$. Since Z is the fundamental cycle, we have $(Z \cdot E_i) \leq 0$ for all i , so $(Z \cdot D) \leq 0$. This implies $D = 0$. $\square$

Hence, we can calculate the highest root divisor by the algorithm of Laufer given in the preceding section. The following proposition gives an algorithm to construct all elements of $R(E)$ from Z by using B :

Theorem 3.4. — *Let $R^+(E) = \{Y_0, \dots, Y_k\}$ with $Y_k = Z$. Then, for each $j = 0, \dots, k-1$, there exists an element Y_t in $R^+(E)$ such that $(Y_t \cdot E_i) = k_i < 0$ and $Y_j = Y_t + k_i E_i$ for some i . Inversely, for each E_i in B such that $(Y_t \cdot E_i) = k_i < 0$, $Y_t + k_i E_i$ is a root divisor in $R(E)$.*

Proof. — The existence of at least one irreducible component E_i in each Y_j such that $(Y_j \cdot E_i) < 0$ is due to negative definiteness of the intersection matrix. Then, theorem follows from the fact that $(Y_t + k_i E_i) \cdot (Y_t + k_i E_i) = -2$. $\square$