

THE CHERN NUMBERS OF THE NORMALIZATION OF AN ALGEBRAIC THREEFOLD WITH ORDINARY SINGULARITIES

by

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Abstract. — By a classical formula due to Enriques, the Chern numbers of the non-singular normalization X of an algebraic surface S with ordinary singularities in $\mathbb{P}^3(\mathbb{C})$ are given by $\int_X c_1^2 = n(n-4)^2 - (3n-16)m + 3t - \gamma$, $\int_X c_2 = n(n^2 - 4n + 6) - (3n-8)m + 3t - 2\gamma$, where n = the degree of S , m = the degree of the double curve (singular locus) D_S of S , t = the cardinal number of the triple points of S , and γ = the cardinal number of the cuspidal points of S . In this article we shall give similar formulas for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ (Theorem 1.15, Theorem 2.1, Theorem 3.2). As a by-product, we obtain a numerical formula for the Euler-Poincaré characteristic $\chi(X, \mathcal{T}_X)$ with coefficient in the sheaf $\mathcal{T}_X$ of holomorphic vector fields on the non-singular normalization X of $\overline{X}$ (Theorem 4.1).

Résumé (Les nombres de Chern de la normalisée d’une variété algébrique de dimension 3 à points singuliers ordinaires)

Par une formule classique due à Enriques, les nombres de Chern de la normalisation non singulière X de la surface algébrique S avec singularités ordinaires dans $\mathbb{P}^3(\mathbb{C})$ sont donnés par $\int_X c_1^2 = n(n-4)^2 - (3n-16)m + 3t - \gamma$, $\int_X c_2 = n(n^2 - 4n + 6) - (3n-8)m + 3t - 2\gamma$, où n est le degré de S , m est le degré de la courbe double (lieu singulier) D_S de S , t est le nombre de points triples de S , et γ est le nombre de points cuspidaux de S . Dans cet article nous donnons des formules similaires pour une “threefold” algébrique $\overline{X}$ avec singularités ordinaires dans $\mathbb{P}^4(\mathbb{C})$ (Théorème 1.15, Théorème 2.1, Théorème 3.2). Comme application, nous obtenons une formule numérique pour la caractéristique d’Euler-Poincaré $\chi(X, \mathcal{T}_X)$ à coefficients dans le faisceau $\mathcal{T}_X$ de champs de vecteurs holomorphes de la normalisation non singulière X de $\overline{X}$ (Théorème 4.1).

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Introduction

An irreducible hypersurface $\overline{X}$ in the complex projective 4-space $\mathbb{P}^4(\mathbb{C})$ is called an *algebraic threefold with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 4-space $\mathbb{C}^4$ at every point of $\overline{X}$:

$$(0.1) \quad \left\{ \begin{array}{ll} \text{(i)} & w = 0 \quad (\text{simple point}) \\ \text{(ii)} & zw = 0 \quad (\text{ordinary double point}) \\ \text{(iii)} & yzw = 0 \quad (\text{ordinary triple point}) \\ \text{(iv)} & xyzw = 0 \quad (\text{ordinary quadruple point}) \\ \text{(v)} & xy^2 - z^2 = 0 \quad (\text{cuspidal point}) \\ \text{(vi)} & w(xy^2 - z^2) = 0 \quad (\text{stationary point}) \end{array} \right.$$

where (x, y, z, w) is the coordinate on $\mathbb{C}^4$. These singularities arise if we project a non-singular threefold embedded in a sufficiently higher dimensional complex projective space to its four dimensional linear subspace by a *generic* linear projection ([**R**]), though the singularities (iv) and (vi) above do not occur in the surface case. This fact can also be proved by use of the classification theory of multi-germs of *locally stable* holomorphic maps ([**M-3**], [**T-1**]). Indeed, in the threefold case, the pair of dimensions of the source and target manifolds belongs to the so-called *nice range* ([**M-2**]). Hence the multi-germ of a *generic* linear projection at the inverse image of any point of $\overline{X}$ is *stable*, i.e., stable under small deformations ([**M-4**]).

In [**T-2**] we have proved, for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ which is free from quadruple points, a formula expressing the Euler number $\chi(X)$ of the non-singular normalization X of $\overline{X}$ in terms of numerical characteristics of $\overline{X}$ and its singular loci. Note that, by the Gauss-Bonnet formula, the Euler number $\chi(X)$ is equal to the Chern number $\int_X c_3$, where c_3 denotes the top Chern class of X . In §1 we shall extend this formula to the general case where $\overline{X}$ admits quadruple points. In this general case, we need to blow up $\overline{X}$ twice. First, along the quadruple point locus, and secondly, along the triple point locus. It turns out that the existence of quadruple points adds only the term $4\#\Sigma\overline{q}$ to the formula, where $\#\Sigma\overline{q}$ denotes the cardinal number of the quadruple point locus $\Sigma\overline{q}$. Using Fulton-MacPherson's intersection theory, especially, the *excess intersection formula* ([**F**], Theorem 6.3, p.102), the *blow-up formula* (ibid., Theorem 6.7, p.116), the *double point formula* (ibid., Theorem 9.3, p.166) and the *ramification formula* (ibid., Example 3.2.20, p.62), we compute the *push-forwards* $f_*[D]^2$ and $f_*[D]^3$ for D the inverse image of the singular locus of $\overline{X}$ by the normalization map in order to know the *Segre classes* $s(\overline{J}, \overline{X})_i$ ($0 \leq i \leq 2$) of the singular subscheme $\overline{J}$ defined by the Jacobian ideal of $\overline{X}$.

In §2 we shall give a formula for the Chern number $\int_X c_1^3 = -[K_X]^3$, where $[K_X]$ is the canonical class of X . The expressions for $f_*[D]^2$ and $f_*[D]^3$ obtained in §1 enable us to compute it, because $[K_X] = f^*[\overline{X} + K_Y] - [D]$ by the *double point formula*, where

K_Y is the canonical divisor of $\mathbb{P}^4(\mathbb{C})$. In §3 we shall give a formula for the Chern number $\int_X c_1 c_2$. In fact, we shall calculate the Euler-Poincaré characteristic $\chi(X, K_X)$ with coefficient in the canonical line bundle of X , which is equal to $-(1/24) \int_X c_1 c_2$ by the Riemann-Roch theorem. In §4, as a by-product, we shall give a numerical formula for the Euler-Poincaré characteristic $\chi(X, \mathcal{T}_X)$ with coefficient in the sheaf $\mathcal{T}_X$ of holomorphic tangent vector fields on X .

Notation and Terminology

Throughout this article we fix the notation as follows:

$Y := \mathbb{P}^4(\mathbb{C})$: the complex projective 4-space,
 $\overline{X}$: an algebraic threefold with ordinary singularities in Y ,
 $\overline{J}$: the singular subscheme of $\overline{X}$ defined by the Jacobian ideal of $\overline{X}$,
 $\overline{D}$: the singular locus of $\overline{X}$,
 $\overline{T}$: the triple point locus of $\overline{X}$, which is equal to the singular locus of $\overline{D}$,
 $\overline{C}$: the cuspidal point locus of $\overline{X}$, precisely, its closure, since we always consider $\overline{C}$ contains the stationary points,
 $\Sigma\overline{q}$: the quadruple point locus of $\overline{X}$,
 $\Sigma\overline{s}$: the stationary point locus of $\overline{X}$,
 $n_{\overline{X}} : X \rightarrow \overline{X}$: the normalization of $\overline{X}$,
 $f : X \rightarrow Y$: the composite of the normalization map $n_{\overline{X}}$ and the inclusion $\overline{\iota} : \overline{X} \hookrightarrow Y$,
 J : the scheme-theoretic inverse of $\overline{J}$ by f ,
 D, T, C and Σq : the inverse images of $\overline{D}, \overline{T}, \overline{C}$ and $\Sigma\overline{q}$ by f , respectively,
 $\Sigma s = T \cap C$: the intersection of T and C .

We put

$$n := \deg \overline{X} \text{ (the degree of } \overline{X} \text{ in } \mathbb{P}^4(\mathbb{C})\text{)}, \quad m := \deg \overline{D}, \quad t := \deg \overline{T}, \quad \gamma := \deg \overline{C}.$$

Note that $\overline{T}$ and $\overline{C}$ are non-singular curves, intersecting transversely at $\Sigma\overline{s}$, and that the normalization X of $\overline{X}$ is also non-singular. Calculating by use of local coordinates, we can easily see the following:

- (i) J contains D , and the *residual scheme* to D in J is the reduced scheme C , i.e., $\mathcal{I}_J = \mathcal{I}_D \otimes_{\mathcal{I}_X} \mathcal{I}_C$, where $\mathcal{I}_J, \mathcal{I}_D, \mathcal{I}_C$ are the ideal sheaves of J, D and C , respectively (cf. [F], Definition 9.2.1, p. 160);
- (ii) D is a surface with ordinary singularities, whose singular locus is T ,
- (iii) D is the *double point* locus of the map $f : X \rightarrow Y$, i.e., the closure of $\{q \in X \mid \#f^{-1}(f(q)) \geq 2\}$;
- (iv) the map $f|_D : D \rightarrow \overline{D}$ is generically two to one, simply ramified at C ;
- (v) the map $f|_T : T \rightarrow \overline{T}$ is generically three to one, simply ramified at Σs .

Furthermore, we need the following diagram consisting of two fiber squares:

$$(0.2) \quad \begin{array}{ccc} X'' & \xrightarrow{f''} & Y'' \\ \tau_{T'} \downarrow & & \downarrow \sigma_{\overline{T}'} \\ X' & \xrightarrow{f'} & Y' \\ \tau_{\Sigma q} \downarrow & & \downarrow \sigma_{\Sigma \overline{q}} \\ X & \xrightarrow{f} & Y, \end{array}$$

which is defined as follows:

$\sigma_{\Sigma \overline{q}} : Y' \rightarrow Y$: the blowing-up of Y along the quadruple point locus $\Sigma \overline{q}$ of $\overline{X}$,

$\overline{X}'$: the proper inverse image of $\overline{X}$ by $\sigma_{\Sigma \overline{q}}$,

$X' := X \times_{\overline{X}} \overline{X}'$: the fiber product of X and $\overline{X}'$ over $\overline{X}$,

$n_{\overline{X}'} : X' \rightarrow \overline{X}'$: the projection to the second factor of $X \times_{\overline{X}} \overline{X}'$, which is nothing but the normalization of $\overline{X}'$,

$f' : X' \rightarrow Y'$: the composite of the normalization map $n_{\overline{X}'}$ and the inclusion

$\iota' : \overline{X}' \hookrightarrow Y'$,

Σq : the inverse image of the quadruple point locus $\Sigma \overline{q}$ of $\overline{X}$ by f ,

$\tau_{\Sigma q} : X' \rightarrow X$: the projection to the first factor of $X \times_{\overline{X}} \overline{X}'$, which is nothing but the blowing-up of X along Σq ,

$\overline{D}', \overline{T}', \overline{C}'$ and $\Sigma \overline{s}'$: the proper inverse images of $\overline{D}, \overline{T}, \overline{C}$ and $\Sigma \overline{s}$ by $\sigma_{\Sigma \overline{q}}$, respectively.

D', T' and C' : the proper inverse images of D, T and C by $\tau_{\Sigma q}$, which are equal to the inverse images of $\overline{D}', \overline{T}'$ and $\overline{C}'$ by f' , respectively,

$\Sigma s'$: the inverse image of Σs by $\tau_{\Sigma q}$, which is equal to $T' \cap C'$,

$\sigma_{\overline{T}'} : Y'' \rightarrow Y'$: the blowing-up of Y' along $\overline{T}'$,

$\overline{X}''$: the proper inverse image of $\overline{X}'$ by $\sigma_{\overline{T}'}$,

$X'' := X' \times_{\overline{X}'} \overline{X}''$: the fiber product of X' and $\overline{X}''$ over $\overline{X}'$,

$n_{\overline{X}''} : X'' \rightarrow \overline{X}''$: the projection to the second factor of $X' \times_{\overline{X}'} \overline{X}''$, which is nothing but the normalization of $\overline{X}''$

$f'' : X'' \rightarrow Y''$: the composite of the normalization map $n_{\overline{X}''}$ and the inclusion

$\iota'' : \overline{X}'' \hookrightarrow Y''$,

$\tau_{T'} : X'' \rightarrow X'$: the projection to the first factor of $X' \times_{\overline{X}'} \overline{X}''$, which is nothing but the blowing-up of X' along T' ,

$\overline{D}'', \overline{T}'', \overline{C}''$ and $\Sigma \overline{s}''$: the proper inverse images of $\overline{D}', \overline{T}', \overline{C}'$ and $\Sigma \overline{s}'$ by $\sigma_{\overline{T}'}$, respectively,

D'', T'' and C'' : the proper inverse images of D', T' and C' by $\tau_{T'}$, which are equal to the inverse images of $\overline{D}'', \overline{T}''$ and $\overline{C}''$ by f'' , respectively,

$\Sigma s''$: the inverse image of $\Sigma s'$ by $\tau_{T'}$, which is equal to $T'' \cap C''$.

We also use the following notation throughout this article:

- $[\alpha]$: the rational equivalence class of an algebraic cycle α ,
- $\alpha \cdot \beta$: the intersection class of two algebraic cycle classes $[\alpha]$ and $[\beta]$.

Finally, we give the definitions of *regular embeddings* and *local complete intersection morphisms* of schemes.

Definition 0.1. — We say a closed embedding $\iota : X \rightarrow Y$ of schemes is a *regular embedding of codimension d* if every point in X has an affine neighborhood U in Y , such that if A is the coordinate ring of U , I the ideal of A defining X , then I is generated by a regular sequence of length d .

If this is the case, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of X in Y , is a locally free sheaf of rank d . The *normal bundle* to X in Y , denoted by $N_X Y$, is the vector bundle on X whose sheaf of sections is dual to $\mathcal{I}/\mathcal{I}^2$. Note that the normal bundle $N_X Y$ is canonically isomorphic to the normal cone $C_X Y$ for a (closed) regular embedding $\iota : X \rightarrow Y$ since the canonical map from $\text{Sym}(\mathcal{I}/\mathcal{I}^2)$ to $S^\cdot := \sum_{k=0}^{\infty} \mathcal{I}^k / \mathcal{I}^{k+1}$ is an isomorphism (cf. [F], Appendix B, B.7).

Definition 0.2. — A morphism $f : X \rightarrow Y$ is called a *local complete intersection morphism of codimension d* if f factors into a (closed) regular embedding $\iota : X \rightarrow P$ of some constant codimension e , followed by a smooth morphism $p : P \rightarrow Y$ of constant relative dimension $d + e$.

1. The computation of $\int_X c_3$

In [T-2] we have proved, for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ which is free from quadruple points, a formula expressing the Euler number $\chi(X)$ of the non-singular normalization X of $\overline{X}$ in terms of numerical characteristics of $\overline{X}$ and its singular loci. We recall its proof briefly. We have first proved the following:

Theorem 1.1 ([T-2], Theorem 2.1). — We have a linear pencil $\overline{\mathcal{L}} := \bigcup_{\lambda \in \mathbb{P}^1} \overline{X}_\lambda$ on $\overline{X}$, consisting of hyperplane sections $\overline{X}_\lambda$ of $\overline{X}$ in $\mathbb{P}^4(\mathbb{C})$, whose pull-back $\mathcal{L} := \bigcup_{\lambda \in \mathbb{P}^1} X_\lambda$ to X by the normalization map $f : X \rightarrow \overline{X}$ has the following properties: There exists a finite set $\{\lambda_1, \dots, \lambda_c\}$ of points of $\mathbb{P}^1$ such that

- (i) X_λ is non-singular for λ with $\lambda \neq \lambda_i$ ($1 \leq i \leq c$), and
- (ii) X_{λ_i} is a surface with only one isolated ordinary double point which is contained in $X \setminus f^{-1}(\overline{C}_\infty)$ for any i with $1 \leq i \leq c$,

where c is the class of $\overline{X}$, i.e., the degree of the top polar class $[M_3]$ of $\overline{X}$ in $\mathbb{P}^4(\mathbb{C})$ (cf. [P]), and $\overline{C}_\infty$ the base point locus of the linear pencil $\overline{\mathcal{L}}$, which is an irreducible curve with m ($= \deg \overline{D}$) ordinary double points in $\mathbb{P}^2(\mathbb{C})$ whose degree is equal to n ($= \deg \overline{X}$).