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GENERALIZED GINZBURG-CHERN CLASSES

by

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***Abstract.* —** For a morphism $f : X \to Y$ with Y being nonsingular, the Ginzburg-Chern class of a constructible function α on the source variety X is defined to be the Chern-Schwartz-MacPherson class of the constructible function α followed by capping with the pull-back of the Segre class of the target variety Y. In this paper we give some generalizations of the Ginzburg-Chern class even when the target variety Y is singular and discuss some properties of them.

***Résumé* (Classes de Ginzburg-Chern généralisées). —** Pour un morphisme algébrique $f : X \to Y$ où la variété Y est non singulière, la classe de Ginzburg-Chern de la fonction constructible α sur la variété source X est définie comme la classe de Chern-Schwartz-MacPherson de la fonction constructible α suivi du cap-produit par l'image réciproque de la classe de Segre de la variété but Y. Dans cet article nous donnons quelques généralisations de la classe de Ginzburg-Chern y compris lorsque la variété but Y est singulière et nous en discutons quelques propriétés.

1. Introduction

In [**G1**] Ginzburg introduced a certain homomorphism from the abelian group of Lagrangian cycles to the Borel-Moore homology group

$$c^{\mathrm{biv}} : L(X_1 \times X_2) \longrightarrow H_*(X_1 \times X_2),$$

which he called a bivariant Chern class. The construction or definition of the homomorphism c^{biv} given in [**G1**] is not direct, but in his survey article [**G2**] he gives an explicit description of it. It assigns to a Lagrangian cycle associated to a subvariety $Y \subset X_1 \times X_2$ the *relative Chern-Mather class of the fibers of the projection*

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$p_Y : Y \to X_2$. The projection p_Y is the restriction of the projection $p_2 : X_1 \times X_2 \to X_2$ to the subvariety Y. Let $\nu : \widehat{Y} \to Y$ be the Nash blow-up and $\widehat{TY}$ the tautological Nash tangent bundle over $\widehat{Y}$. Then the above relative Chern-Mather class is defined by

$$c^{\mathrm{biv}}(\Lambda_Y) := i_{Y*}\nu_*\Big(c(\widehat{TY} - \nu^* p_Y^* TX_2) \cap [\widehat{Y}]\Big)$$

where $i_Y : Y \to X_1 \times X_2$ is the inclusion. Then it follows from the projection formula and from $p_Y = p_2 \circ i_Y$ that

$$\begin{aligned} c^{\mathrm{biv}}(\Lambda_Y) &= i_{Y*}\Big(\frac{1}{p_Y^* c(TX_2)} \cap c^M(Y)\Big) \\ &= p_2^* s(TX_2) \cap i_{Y*} c^M(Y). \end{aligned}$$

Here $s(TX_2)$ denotes the Segre class of the tangent bundle TX_2.

Since the Chern-Schwartz-MacPherson class ([**BS**], [**M**], [**Sw1**], [**Sw2**] etc.) is a linear combination of Chern-Mather classes, the above homomorphism c^{biv} can be defined for any morphism $\pi : X \to Y$ from a possibly singular variety X to a smooth variety Y and for any constructible function on the target variety X. Namely we can define the following homomorphism

$$\pi^* s(TY) \cap c_* : F(X) \longrightarrow H_*(X; \mathbf{Z})$$

where $c_* : F(X) \to H_*(X; \mathbf{Z})$ is the usual Chern-Schwartz-MacPherson class transformation. This "twisted" Chern-Schwartz-MacPherson class shall be called the *Ginzburg-Chern class*.

On the other hand, in [**Y3**] we showed that the bivariant Chern class ([**Br**], [**FM**]) for any morphism with nonsingular target variety necessarily has to be the Ginzburg-Chern class. To be more precise, if there exists a bivariant Chern class $\gamma : \mathbf{F} \to \mathbf{H}$ from the Fulton-MacPherson bivariant theory of constructible functions to the Fulton-MacPherson bivariant homology theory, then for any morphism $f : X \to Y$ with Y being nonsingular and any bivariant constructible function $\alpha \in \mathbf{F}(X \to Y)$ the following holds

$$\gamma_f(\alpha) = f^* s(TY) \cap c_*(\alpha),$$

where $\gamma_f : \mathbf{F}(X \xrightarrow{f} Y) \to \mathbf{H}(X \xrightarrow{f} Y)$.

Quickly speaking, this theorem follows from the simple observation that for $\alpha \in \mathbf{F}(X \to Y) \subset F(X)$ we have

$$c_*(\alpha) = \gamma_f(\alpha) \bullet c_*(Y),$$

where $\bullet$ on the right-hand-side is the bivariant product. Thus a naïve solution for $\gamma_f(\alpha)$ is the following "quotient"

$$\gamma_f(\alpha) = \text{``}\frac{c_*(\alpha)}{c_*(Y)}\text{''}.$$

It turns out that in the case when the target variety Y is nonsingular this "quotient" is well-defined and it is nothing but

$$“\frac{c_*(\alpha)}{c_*(Y)}” = \frac{c_*(\alpha)}{f^*c(TY)} = f^*s(TY) \cap c_*(\alpha).$$

From now on the Ginzburg-Chern class of α shall be denoted by $\gamma^{\mathrm{Gin}}(\alpha)$ or $\gamma_f^{\mathrm{Gin}}(\alpha)$ emphasizing the morphism f.

As one sees, for the definition of the Ginzburg-Chern class the nonsingularity of the target variety Y is clearly essential. In this paper, we put aside the bivariant-theoretic aspect of the Ginzburg-Chern class ([**Y4**], [**Y5**], [**Y6**]) and, using Nash blow-ups and also resolutions of singularities we introduce reasonably modified versions of the Ginzburg-Chern class, even when the target variety is arbitrarily singular. We discuss some properties of them and in particular we obtain some results concerning the convolution product of them.

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2. Generalized Ginzburg-Chern classes

The Ginzburg-Chern class is a unique natural transformation satisfying a certain normalization in the following sense:

Theorem 2.1 **([Y2**, Theorem (2.1)**])**. — *For the category of Y-varieties, i.e., morphisms $\pi : X \to Y$, with Y being a nonsingular variety, $\gamma_\pi^{\mathrm{Gin}} : F(X) \to H_*(X; \mathbf{Z})$ is the unique natural transformation from the constructible functions to the homology theory such that for a smooth variety X we have*

$$\gamma_\pi^{\mathrm{Gin}}(1\!\!1_X) = c(T_\pi) \cap [X], \tag{2.2}$$

where $T_\pi := TX - \pi^*TY$ is the relative virtual tangent bundle. Namely, for any commutative diagram

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & Y & \end{tikzcd}$$

where f is proper, we have the following commutaive diagram

$$\begin{tikzcd} F(X_1) \arrow[r, "f_*"] \arrow[d, "\gamma^{\mathrm{Gin}}_{\pi_1}"'] & F(X_2) \arrow[d, "\gamma^{\mathrm{Gin}}_{\pi_2}"] \\ H_*(X_1) \arrow[r, "f_*"'] & H_*(X_2). \end{tikzcd}$$

A natural question or problem on the Ginzburg-Chern class is whether or not one can extend it to the case when the target variety Y is singular and we want to see if a theorem similar to the above one holds.

Suppose that Y is singular and we consider the Nash blow-up $\nu : \widehat{Y} \to Y$ and the following fiber square

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\nu}"] \arrow[d, "\widehat{\pi}"'] & X \arrow[d, "\pi"] \\ \widehat{Y} \arrow[r, "\nu"'] & Y. \end{tikzcd}$$

Then we define the homomorphism

$$\widehat{\gamma}^{\mathrm{Gin}}_{\pi} : F(X) \longrightarrow H_*(X; \mathbf{Z})$$

by

$$\widehat{\gamma}^{\mathrm{Gin}}_{\pi} := \widehat{\nu}_*\Big(\widehat{\pi}^* s(\widehat{TY}) \cap c_*(\widehat{\nu}^* \alpha)\Big). \tag{2.3}$$

This class shall be called *a Nash-type Ginzburg-Chern class*, abusing words. Then we have the following theorem:

Theorem 2.4. — *Let Y be a possibly singular variety. Then, for any commutative diagram*

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & Y & \end{tikzcd}$$

where f is proper, we have the following commutative diagram

$$\begin{tikzcd} F(X_1) \arrow[r, "f_*"] \arrow[d, "\widehat{\gamma}^{\mathrm{Gin}}_{\pi_1}"'] & F(X_2) \arrow[d, "\widehat{\gamma}^{\mathrm{Gin}}_{\pi_2}"] \\ H_*(X_1) \arrow[r, "f_*"'] & H_*(X_2). \end{tikzcd}$$

Proof. — First we recall the following fact ([**Er**, Proposition 3.5], [**FM**, Axiom (A_{23})]): for any fiber square

$$\begin{tikzcd} W' \arrow[r, "g'"] \arrow[d, "h'"'] & W \arrow[d, "h"] \\ Z' \arrow[r, "g"] & Z, \end{tikzcd}$$

the following diagram commutes

$$\begin{tikzcd} F(W) \arrow[r, "g'^*"] \arrow[d, "h'_*"'] & F(W') \arrow[d, "h_*"] \\ F(Z) \arrow[r, "g^*"] & F(Z'). \end{tikzcd}$$

Now we have the following commutative diagrams:

$$\begin{tikzcd} \widehat{X}_1 \arrow[rrr, "\widehat{\nu}_1"] \arrow[dr, "\widehat{f}"] \arrow[dd, "\widehat{\pi}_1"'] & & & X_1 \arrow[dl, "f"'] \arrow[dd, "\pi_1"] \\ & \widehat{X}_2 \arrow[r, "\widehat{\nu}_2"] \arrow[dl, "\widehat{\pi}_2"] & X_2 \arrow[dr, "\pi_2"'] & \\ \widehat{Y} \arrow[rrr, "\nu"] & & & Y. \end{tikzcd}$$

Then by definition we have

$$\begin{aligned} \widehat{\gamma}^{\mathrm{Gin}}_{\pi_2}(f_*\alpha) &= \widehat{\nu}_{2*}\Big(\widehat{\pi}_2^* s(\widehat{TY}) \cap c_*(\widehat{\nu}_2^* f_*\alpha)\Big) \\ &= \widehat{\nu}_{2*}\Big(\widehat{\pi}_2^* s(\widehat{TY}) \cap c_*(\widehat{f}_*\widehat{\nu}_1^*\alpha)\Big) \\ &= \widehat{\nu}_{2*}\Big(\widehat{\pi}_2^* s(\widehat{TY}) \cap \widehat{f}_* c_*(\widehat{\nu}_1^*\alpha)\Big) \\ &= \widehat{\nu}_{2*}\widehat{f}_*\Big(\widehat{f}^*\widehat{\pi}_2^* s(\widehat{TY}) \cap c_*(\widehat{\nu}_1^*\alpha)\Big) \\ &= f_*\widehat{\nu}_{1*}\Big(\widehat{\pi}_1^* s(\widehat{TY}) \cap c_*(\widehat{\nu}_1^*\alpha)\Big) \\ &= f_*\widehat{\gamma}^{\mathrm{Gin}}_{\pi_1}(\alpha). \end{aligned}$$

□

Motivated by the definition of the Nash-type Ginzburg-Chern class, we give another modification of the Ginzburg-Chern class via a resolution of singularities. Let $\rho : \widetilde{Y} \to$