

EXPLICIT UPPER BOUNDS FOR THE RESIDUES AT $s = 1$ OF THE DEDEKIND ZETA FUNCTIONS OF SOME TOTALLY REAL NUMBER FIELDS

by

Stéphane R. Louboutin

Abstract. — We give an explicit upper bound for the residue at $s = 1$ of the Dedekind zeta function of a totally real number field K for which $\zeta_K(s)/\zeta(s)$ is entire. Notice that this is conjecturally always the case, and that it holds true if $K/\mathbf{Q}$ is normal or if K is cubic.

Résumé (Bornes supérieures explicites pour les résidus en $s = 1$ des fonctions zêta de Dedekind de corps de nombres totalement réels)

Nous donnons une borne supérieure explicite pour le résidu en $s = 1$ de la fonction zêta de Dedekind d'un corps de nombres K totalement réel pour lequel $\zeta_K(s)/\zeta(s)$ est entière. On remarque que c'est conjecturalement toujours le cas, et que c'est vrai si $K/\mathbf{Q}$ est normale ou si K est cubique.

1. Introduction

Let d_K and $\zeta_K(s)$ denote the absolute value of the discriminant and the Dedekind zeta function of a number field K of degree $m > 1$. It is important to have explicit upper bounds for the residue at $s = 1$ of $\zeta_K(s)$. As for the best general such bounds, we have (see [Lou01, Theorem 1]):

$$\mathrm{Res}_{s=1}(\zeta_K(s)) \leq \left(\frac{e \log d_K}{2(m-1)} \right)^{m-1}.$$

However, for some totally real number fields an improvement on this bound is known (see [BL] and [Oka] for applications):

Theorem 1 (See [Lou01, Theorem 2]). — *Let K range over a family of totally real number fields of a given degree $m \geq 3$ for which $\zeta_K(s)/\zeta(s)$ is entire. There exists C_m*

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(computable) such that $d_K \geq C_m$ implies

$$\text{Res}_{s=1}(\zeta_K(s)) \leq \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!} \leq \frac{1}{\sqrt{2\pi(m-1)}} \left(\frac{e \log d_K}{2(m-1)} \right)^{m-1}.$$

Moreover, for any non-normal totally real cubic field K we have the slightly better bound

$$\text{Res}_{s=1}(\zeta_K(s)) \leq \frac{1}{8}(\log d_K - \kappa)^2$$

where $\kappa := 2 \log(4\pi) - 2 - 2\gamma = 1.90761 \dots$.

Remark 2. — If $K/\mathbf{Q}$ is normal or if K is cubic, then $\zeta_K(s)/\zeta(s)$ is entire.

We will simplify our previous proof of Theorem 1 (by improving those of [Lou98, Theorem 5] and [Lou01, Theorem 2]) and we will give explicit constants C_m for which Theorem 1 holds true:

Theorem 3. — There exists $C > 0$ (effective) such that for any totally real number field K of degree $m \geq 3$ and root discriminant $\rho_K := d_K^{1/m} \geq C^m$ we have

$$\text{Res}_{s=1}(\zeta_K(s)) \leq \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!},$$

provided that $\zeta_K(s)/\zeta(s)$ is entire. Moreover, $C = 3309$ will do for m large enough.

This result is not the one we would have wished to prove. It would indeed have been much more satisfactory to prove that there exists $C > 0$ (effective) such that this bound is valid for such totally real number fields K of root discriminants $\rho_K \geq C$ large enough. It would have been even more satisfactory to prove that this constant C is small enough to obtain that our bound is valid for all totally real number fields K for which $\zeta_K(s)/\zeta(s)$ is entire (e.g., see [Was, Page 224] for explicit lower bounds on root discriminants of totally real number fields K). Let us finally point out that, in the case that $K/\mathbf{Q}$ is abelian, we have an even better bound (see [Lou01, Corollary 8] and use [Ram, Corollary 1]):

$$\text{Res}_{s=1}(\zeta_K(s)) \leq \left(\frac{\log d_K}{2(m-1)} \right)^{m-1}.$$

2. Proof of Theorem 1

Proposition 4. — Let K be a totally real number field of degree $m \geq 1$, set $d = \sqrt{d_K}$, and assume that $\zeta_K(s)/\zeta(s)$ is entire. Then, $\text{Res}_{s=1}(\zeta_K(s)) \leq \rho_{m-1}(d)$ where

$$(1) \quad \rho_{m-1}(d) := \text{Res}_{s=1} \left\{ s \mapsto \left(\pi^{-s/2} \Gamma(s/2) \zeta(s) \right)^{m-1} \left(\frac{1}{s} + \frac{1}{s-1} \right) (d^{s-1} + d^{-s}) \right\}.$$

Proof. — To begin with, we set some notation: if K is a totally real number field of degree $m \geq 1$, we set $A_K = \sqrt{d_K/\pi^m}$ and $F_K(s) = A_K^s \Gamma^m(s/2) \zeta_K(s)$. Hence, $F_K(s)$ is meromorphic, with only two poles, at $s = 1$ and $s = 0$, both simple, and it satisfies the functional equation $F_K(1-s) = F_K(s)$.

We then set $F_{K/\mathbf{Q}}(s) = F_K(s)/F_{\mathbf{Q}}(s)$, which under our assumption is entire, and satisfies the functional equation $F_{K/\mathbf{Q}}(1-s) = F_{K/\mathbf{Q}}(s)$, and $A_{K/\mathbf{Q}} := A_K/A_{\mathbf{Q}} = \sqrt{d_K/\pi^{m-1}}$. Notice that $F_{K/\mathbf{Q}}(1) = \sqrt{d_K} \operatorname{Res}_{s=1}(\zeta_K(s))$. Let

$$(2) \quad S_{K/\mathbf{Q}}(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_{K/\mathbf{Q}}(s) x^{-s} ds \quad (c > 1 \text{ and } x > 0)$$

denote the Mellin transform of $F_{K/\mathbf{Q}}(s)$. Since $F_{K/\mathbf{Q}}(s)$ is entire, it follows that $S_{K/\mathbf{Q}}(x)$ satisfies the functional equation

$$(3) \quad S_{K/\mathbf{Q}}(x) = \frac{1}{x} S_{K/\mathbf{Q}}\left(\frac{1}{x}\right)$$

(shift the vertical line of integration $\Re(s) = c > 1$ in (2) leftwards to the vertical line of integration $\Re(s) = 1-c < 0$, then use the functional equation $F_{K/\mathbf{Q}}(1-s) = F_{K/\mathbf{Q}}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$), and

$$(4) \quad F_{K/\mathbf{Q}}(s) = \int_0^\infty S_{K/\mathbf{Q}}(x) x^s \frac{dx}{x} = \int_1^\infty S_{K/\mathbf{Q}}(x) (x^s + x^{1-s}) \frac{dx}{x}$$

is the inverse Mellin transform of $S_{K/\mathbf{Q}}(x)$.

Now, set

$$(5) \quad \begin{aligned} F_{m-1}(s) &= F_{\mathbf{Q}}^{m-1}(s) = (\pi^{-s/2} \Gamma(s/2) \zeta(s))^{m-1}, \\ A_{m-1} &= A_{\mathbf{Q}}^{m-1} = \pi^{-(m-1)/2} \end{aligned}$$

and let

$$(6) \quad S_{m-1}(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_{m-1}(s) x^{-s} ds \quad (c > 1 \text{ and } x > 0)$$

denote the Mellin transform of $F_{m-1}(s)$. Here, $F_{m-1}(s)$ has two poles, at $s = 1$ and $s = 0$, the functional equation $F_{m-1}(1-s) = F_{m-1}(s)$ yields

$$\operatorname{Res}_{s=0}(F_{m-1}(s)x^{-s}) = -\operatorname{Res}_{s=1}(F_{m-1}(s)x^{s-1})$$

and

$$(7) \quad S_{m-1}(x) = \operatorname{Res}_{s=1}\{F_{m-1}(s)(x^{-s} - x^{s-1})\} + \frac{1}{x} S_{m-1}\left(\frac{1}{x}\right)$$

(shift the vertical line of integration $\Re(s) = c > 1$ in (6) leftwards to the vertical line of integration $\Re(s) = 1-c < 0$, notice that you pick up residues at $s = 1$ and $s = 0$, then use the functional equation $F_{m-1}(1-s) = F_{m-1}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$). Finally, we set

$$H_{m-1}(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma^{m-1}(s/2) x^{-s} ds \quad (c > 1 \text{ and } x > 0).$$

Notice that $0 < H_{m-1}(x)$ for $x > 0$ (see [Lou00, Proof of Theorem 2]⁽¹⁾). Now, write

$$\zeta_K(s)/\zeta(s) = \sum_{n \geq 1} a_{K/\mathbf{Q}}(n)n^{-s}$$

and

$$\zeta^{m-1}(s) = \sum_{n \geq 1} a_{m-1}(n)n^{-s}.$$

Then, $|a_{K/\mathbf{Q}}(n)| \leq a_{m-1}(n)$ for all $n \geq 1$ (see [Lou01, Lemma 26]). Since

$$S_{K/\mathbf{Q}}(x) = \sum_{n \geq 1} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/A_{K/\mathbf{Q}})$$

and

$$0 \leq S_{m-1}(x) = \sum_{n \geq 1} a_{m-1}(n)H_{m-1}(nx/A_{m-1}),$$

we obtain

$$(8) \quad S_{K/\mathbf{Q}}(x) \leq S_{m-1}(x/d) \quad \text{with } d := A_{K/\mathbf{Q}}/A_{m-1} = \sqrt{d_K}.$$

We are now ready to proceed with the proof of Proposition 4. We have

$$\begin{aligned} d \operatorname{Res}_{s=1}(\zeta_K(s)) &= F_{K/\mathbf{Q}}(1) = \int_1^\infty S_{K/\mathbf{Q}}(x) \left(1 + \frac{1}{x}\right) dx \quad (\text{by (4)}) \\ &\leq \int_1^\infty S_{m-1}(x/d) \left(1 + \frac{1}{x}\right) dx \quad (\text{by (8)}) \\ &= \int_{1/d}^\infty S_{m-1}(x) \left(d + \frac{1}{x}\right) dx \\ &= \int_1^\infty S_{m-1}(x) \left(d + \frac{1}{x}\right) dx + \int_1^d \frac{1}{x} S_{m-1}\left(\frac{1}{x}\right) \left(\frac{d}{x} + 1\right) dx \\ &\leq (d+1) \int_1^\infty S_{m-1}(x) \left(1 + \frac{1}{x}\right) dx \\ &\quad - \int_1^d \operatorname{Res}_{s=1}\{F_{m-1}(s)(x^{-s} - x^{s-1})\} \left(\frac{d}{x} + 1\right) dx \\ &(\text{by (7), and for } S_{m-1}(x) \geq 0 \text{ for } x > 0) \end{aligned}$$

⁽¹⁾Notice the misprints in [Lou00, page 273, line 1] and [Lou01, Theorem 20] where one should read

$$(M_1 \star M_2)(x) = \int_0^\infty M_1(x/t)M_2(t)\frac{dt}{t}.$$

$$\begin{aligned}
&= (d+1) \int_1^\infty S_{m-1}(x) \left(1 + \frac{1}{x}\right) dx \\
&\quad - \operatorname{Res}_{s=1} \left\{ F_{m-1}(s) \int_1^d (x^{-s} - x^{s-1}) \left(\frac{d}{x} + 1\right) dx \right\} \\
&\quad \text{(compute these residues as contour integrals along a circle} \\
&\quad \text{of center 1 and of small radius, and use Fubini's theorem)} \\
&= (d+1) \left(\int_1^\infty S_{m-1}(x) \left(1 + \frac{1}{x}\right) dx - \operatorname{Res}_{s=1} \left\{ F_{m-1}(s) \left(\frac{1}{s} + \frac{1}{s-1}\right) \right\} \right) \\
&\quad + \operatorname{Res}_{s=1} \left\{ F_{m-1}(s) (d^s + d^{1-s}) \left(\frac{1}{s} + \frac{1}{s-1}\right) \right\}.
\end{aligned}$$

The desired result now follows from Lemma 5 below. $\square$

Lemma 5. — *Set*

$$G_{m-1}(s) := F_{m-1}(s) \left(\frac{1}{s} + \frac{1}{s-1}\right).$$

Then,

$$I_{m-1} := \int_1^\infty S_{m-1}(x) \left(1 + \frac{1}{x}\right) dx = \operatorname{Res}_{s=1}(G_{m-1}(s)).$$

Proof. — By (6) and Fubini's theorem, we have

$$I_{m-1} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_{m-1}(s) \left(\int_1^\infty (x^{-s} + x^{-s-1}) dx \right) ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G_{m-1}(s) ds$$

The functional equation $G_{m-1}(1-s) = -G_{m-1}(s)$ yields

$$\begin{aligned}
I_{m-1} &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G_{m-1}(s) ds \\
&= \operatorname{Res}_{s=1}(G_{m-1}(s)) + \operatorname{Res}_{s=0}(G_{m-1}(s)) + \frac{1}{2\pi i} \int_{1-c-i\infty}^{1-c+i\infty} G_{m-1}(s) ds \\
&= 2 \operatorname{Res}_{s=1}(G_{m-1}(s)) - I_{m-1},
\end{aligned}$$

from which the desired result follows. $\square$

Let us now complete the proof of Theorem 1. Since

$$(9) \quad \pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s-1} - a + O(s-1),$$

with $a = (\log(4\pi) - \gamma)/2 = 0.97690\dots$, using (1) we obtain

$$\rho_{m-1}(d) = \frac{1}{(m-1)!} \log^{m-1} d - \frac{c_{m-1}}{(m-2)!} \log^{m-2} d + O(\log^{m-3} d)$$

with $c_{m-1} := (m-1)a - 1 > 0$ for $m \geq 3$, and the desired first result follows. In the special case $m = 3$, in writing

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s-1} - a + b(s-1) + O((s-1)^2),$$