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ON THE BILINEAR COMPLEXITY OF THE MULTIPLICATION IN FINITE FIELDS

by

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Abstract. — The aim of this paper is to introduce the bilinear complexity of the multiplication in finite fields and to give a brief exposition of the recent results obtained in this part of algebraic complexity theory. In particular we present the new results obtained using the Chudnovsky-Chudnovsky algorithm and its generalizations.

Résumé **(Sur la complexité bilinéaire de la multiplication dans les corps finis)**
L'objectif de cet article est de présenter la complexité bilinéaire de la multiplication dans les corps finis et de faire un bref tour d'horizon des résultats récents obtenus dans cette partie de la théorie de la complexité algébrique. En particulier, nous présentons les résultats nouveaux qui découlent de l'utilisation de l'algorithme de Chudnovsky-Chudnovsky et de ses généralisations.

1. Introduction

The aim of this paper is to introduce the bilinear complexity of the multiplication in finite fields and to give a brief exposition of the recent results obtained in this part of algebraic complexity theory. The best general reference here is [**8**].

In this section we introduce the problem, we set up notation and terminology and we review some of the standard results on the multiplication of two polynomials modulo a given polynomial.

In section 2, we summarize without proof the algorithm of D.V. Chudnovski and G.V. Chudnovski (*cf.* [**9**]). This algorithm results in the *linearity of the bilinear complexity of the multiplication*. We explain that, in some sense, the algorithm of D.V. Chudnovski and G.V. Chudnovski is not so far from a Fourier Transform. We give also lower and upper asymptotic estimates of the bilinear complexity, due to Shparlinski, Tsfasman, Vladut (*cf.* [**15**]). We present the results obtained by the use of the D.V. Chudnovski and G.V. Chudnovski algorithm with elliptic curves (*cf.* [**14**]).

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In section 3, we introduce a generalization of the D.V. Chudnovski and G.V. Chudnovski algorithm (*cf.* [**6**]), and the recent results we have obtained on the upper bounds for the bilinear complexity of the multiplication. We also describe some towers of algebraic function fields used to obtain the different estimates.

1.1. The bilinear complexity of the multiplication. — Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements where p is a prime number. Let $\mathbb{F}_{q^n}$ be a degree n extension of $\mathbb{F}_q$. The multiplication m in the finite field $\mathbb{F}_{q^n}$ is a bilinear map from $\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ into $\mathbb{F}_{q^n}$, thus it corresponds to a linear map M from the tensor product $\mathbb{F}_{q^n} \otimes \mathbb{F}_{q^n}$ into $\mathbb{F}_{q^n}$. One can also represent M by a tensor $t_M \in \mathbb{F}_{q^n}^* \otimes \mathbb{F}_{q^n}^* \otimes \mathbb{F}_{q^n}$ where $\mathbb{F}_{q^n}^*$ denotes the algebraic dual of $\mathbb{F}_{q^n}$. Each decomposition

$$t_M = \sum_{i=1}^{k} a_i^* \otimes b_i^* \otimes c_i \tag{1}$$

of the tensor t_M, where $a_i^*, b_i^* \in \mathbb{F}_{q^n}^*$ and $c_i \in \mathbb{F}_{q^n}$, brings forth a multiplication algorithm

$$x \cdot y = t_M(x \otimes y) = \sum_{i=1}^{k} a_i^*(x) \otimes b_i^*(x) \otimes c_i.$$

The bilinear complexity of the multiplication in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, denoted by $\mu_q(n)$, is the minimum number of summands in the decomposition (1). Alternatively, we can say that the bilinear complexity of the multiplication is the rank of the tensor t_M (*cf.* [**15**], [**2**]).

1.2. Complexity and bilinear complexity of the multiplication. — Let us remark that the bilinear complexity of the multiplication is far from being the global complexity of the multiplication. If we use the decomposition (1), all the operations involved in the linear part of the computation, namely the computations of $x_i^*(x)$ and $y_i^*(y)$, are not taken in account for the bilinear complexity. But in fact these operations can have a heavy cost. If we take for example the multiplication of polynomials with complex coefficients, and if we use a well fitted Fourier transform, the bilinear complexity is linear, but the complexity of the fast Fourier transforms which constitute the linear part of the algorithm is $O(n \ln(n))$. However, it is suitable to count separately the linear complexity and the bilinear complexity. Indeed, if we want to multiply two variables x and y we have to design a general algorithm of multiplication, but if we want to multiply a given constant a by a variable x, the algorithm can be simpler, because we can adapt the algorithm to the particular value a (think for example to the particular case $a = 1$). In the paper, our purpose is to study the bilinear complexity. No attempt has been made here to develop a study of the linear complexity.

1.3. Old classical results. — Let

$$P(u) = \sum_{i=0}^{n} a_i u^i$$

be a monic irreducible polynomial of degree n with coefficients in a field F. Let

$$R(u) = \sum_{i=0}^{n-1} x_i u^i \quad \text{and} \quad S(u) = \sum_{i=0}^{n-1} y_i u^i$$

be polynomial of degree $\leqslant n-1$ where the coefficients x_i and y_i are indeterminates. As a consequence of a result of Fiduccia and Zalestein (*cf.* [**10**], [**8**] p. 367 prop. 14.47) the bilinear complexity of the multiplication $R(u) \times S(u)$ is $\geqslant 2n-1$. When the field F is infinite, an algorithm reaching exactly this bound was previously given by Toom in [**16**]. Winograd described in [**17**] all the algorithms reaching the bound $2n-1$. Moreover, Winograd proved in [**18**] that up to some transformation every algorithm for computing the coefficients of $R(u) \times S(u) \mod P(u)$ which is of bilinear complexity $2n-1$, necessarily computes the coefficients of $R(u) \times S(u)$, and consequently uses one of the algorithms described in [**17**]. These algorithms use interpolation technics and cannot be performed if the cardinality of the field F is $< 2n-2$. In conclusion we have the following result:

Theorem 1.1. — *If the cardinality of F is $< 2n-2$, every algorithm computing the coefficients of $R(u) \times S(u) \mod P(u)$ has a bilinear complexity $> 2n-1$.*

Applying the results of Winograd and Theorem 1.1 to the multiplication in a finite extension $\mathbb{F}_{q^n}$ of a finite field $\mathbb{F}_q$ we obtain:

Theorem 1.2. — *The bilinear complexity $\mu_q(n)$ of the multiplication in the finite field $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ verifies*

$$\mu_q(n) \geqslant 2n-1,$$

with equality holding if and only if

$$n \leqslant \frac{q}{2} + 1.$$

This result does not give any estimate of an upper bound for $\mu_q(n)$, when n is large. In [**13**], Lempel, Seroussi and Winograd proved that $\mu_q(n)$ has a quasi-linear upper bound. More precisely:

Theorem 1.3. — *The bilinear complexity of the multiplication in the finite field $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ verifies:*

$$\mu_q(n) \leqslant f_q(n) n,$$

where $f_q(n)$ is a very slowly growing function, namely

$$f_q(n) = O(\underbrace{\log_q \log_q \cdots \log_q}_{k \text{ times}}(n))$$

for any $k \geqslant 1$.

2. Interpolation on algebraic curves

We have seen in the previous section that if the number of points of the ground field is too low, we cannot perform the multiplication by the Winograd interpolation method. D.V. and G.V. Chudnowski have designed in [**9**] an algorithm where the interpolation is done on points of an algebraic curve over the groundfield with a sufficient number of rational points. Using this algorithm, D.V. and G.V. Chudnowski proved that *the bilinear complexity of the multiplication in finite extensions of a finite field is linear.*

2.1. Linearity of the bilinear complexity of the multiplication

2.1.1. The D.V Chudnovski and G.V. Chudnovski algorithm. — Let us introduce first the D.V Chudnovski and G.V. Chudnovski theorems proved in [**9**].

Theorem 2.1. — *Let*

- $F/\mathbb{F}_q$ *be an algebraic function field,*
- Q *be a degree* n *place of* $F/\mathbb{F}_q$*,*
- $\mathcal{D}$ *be a divisor of* $F/\mathbb{F}_q$*,*
- $\mathcal{P} = \{P_1, \ldots, P_N\}$ *be a set of places of degree* 1.

We suppose that Q*,* $P_1, \ldots, P_N$ *are not in the support of* $\mathcal{D}$ *and that:*

(a) *The evaluation map*

$$\mathrm{Ev}_Q : \mathcal{L}(\mathcal{D}) \longrightarrow \mathbb{F}_{q^n} \simeq F_Q$$

is onto (where F_Q *is the residue class field of* Q*),*

(b) *the application*

$$\mathrm{Ev}_{\mathcal{P}} : \begin{cases} \mathcal{L}(2\mathcal{D}) \longrightarrow \mathbb{F}_q^N \\ \qquad f \longmapsto (f(P_1), \ldots, f(P_N)) \end{cases}$$

is injective.

Then

$$\mu_q(n) \leqslant N.$$

Sketch of proof. — Let x an y be two elements of $\mathbb{F}_{q^n}$. We know that the residue class field F_Q is isomorphic to $\mathbb{F}_{q^n}$, hence x and y can be considered as element of F_Q. From the condition a), there exist two algebraic functions f and g in $\mathcal{L}(\mathcal{D})$ such that $f(Q) = x$ and $g(Q) = y$. Now we can evaluate f and g on the points $P_1, \ldots, P_N$. In this way we can compute with N bilinear multiplications the evaluation of $h = f \cdot g$ on these points:

$$(h(P_1) \cdots h(P_N)) = (f(P_1)g(P_1), \ldots, f(P_N)g(P_N)).$$

We know that $h \in \mathcal{L}(2\mathcal{D})$, hence, using the condition b) we can find h. Now we can conclude by computing $h(Q)$ which is in fact $f(Q)g(Q) = xy$. The only bilinear computation is the computation of the N products $f(P_i)g(P_i)$. □

Using this algorithm with a good sequence of algebraic function fields, D.V. Chudnovski and G.V. Chudnovski proved the linearity of the bilinear complexity of the multiplication:

Theorem 2.2. — *For any prime power q, there exists a constant C_q such that*

$$\mu_q(n) \leqslant C_q n.$$

2.1.2. Asymptotic bounds. — Shparlinski, Tsfasman, Vladut have given in [**15**] many interesting remarks on the algorithm of D.V. and G.V. Chudnovski. They have linked the algorithm with coding theory, and more precisely with the notion of supercode. They have also obtained in the same paper asymptotic bounds for the bilinear complexity. Following the authors, let us define

$$M_q = \limsup_{k\to\infty} \frac{\mu_q(k)}{k} \quad \text{and} \quad m_q = \liminf_{k\to\infty} \frac{\mu_q(k)}{k}.$$

Let us summarize the estimates given in [**15**]:

(1) $q = 2$

$$3.52 \leqslant m_2 \leqslant 35/6.$$
$$M_2 \leqslant 27.$$

(2) $q \geqslant 9$ is a square

$$2 + \frac{1}{q-1} \leqslant m_q \leqslant 2\left(1 + \frac{1}{\sqrt{q}-2}\right).$$
$$M_q \leqslant 2\left(1 + \frac{1}{\sqrt{q}-2}\right).$$

(3) $q > 2$

$$2 + \frac{1}{q-1} \leqslant m_q \leqslant 3\left(1 + \frac{1}{q-2}\right).$$
$$M_q \leqslant 6\left(1 + \frac{1}{q-2}\right).$$

2.1.3. The use of elliptic curves. — Applying the D.V. and G.V. Chudnovski algorithm with well fitted elliptic curves, Shokrollahi has shown (*cf.* [**14**]) that:

Theorem 2.3. — *The bilinear complexity $\mu_q(n)$ of the multiplication in the finite extension $\mathbb{F}_{q^n}$ of the finite field $\mathbb{F}_q$ is equal to $2n$ for*

$$\frac{1}{2}q + 1 < n < \frac{1}{2}(q + 1 + \varepsilon(q)) \tag{2}$$

where ε is the function defined by:

$$\varepsilon(q) = \begin{cases} \text{greatest integer} \leqslant 2\sqrt{q} \text{ prime to } q, & \text{if } q \text{ is not a perfect square} \\ 2\sqrt{q}, & \text{if } q \text{ is a perfect square.} \end{cases}$$

We do not know if the converse is true. More precisely the question is: suppose that $\mu_q(n) = 2n$, are the inequalities (2) true?