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VARIATION OF PARABOLIC COHOMOLOGY AND POINCARÉ DUALITY

by

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Abstract. — We continue our study of the variation of parabolic cohomology ([**DW**]) and derive an exact formula for the underlying Poincaré duality. As an illustration of our methods, we compute the monodromy of the Picard-Euler system and its invariant Hermitian form, reproving a classical theorem of Picard.

***Résumé* (Variation de la cohomologie parabolique et dualité de Poincaré).** — On continue l'étude de la variation de la cohomologie parabolique commencée dans [**DW**]. En particulier, on donne des formules pour l'accouplement de Poincaré sur la cohomologie parabolique, et on calcule la monodromie du système de Picard-Euler, confirmant un résultat classique de Picard.

Introduction

Let $x_1, \ldots, x_r$ be pairwise distinct points on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and set $U := \mathbb{P}^1(\mathbb{C}) - \{x_1, \ldots, x_r\}$. The Riemann–Hilbert correspondence [**Del70**] is an equivalence between the category of ordinary differential equations with polynomial coefficients and at most regular singularities at the points x_i and the category of local systems of $\mathbb{C}$-vectorspaces on U. The latter are essentially given by an r-tuple of matrices $g_1, \ldots, g_r \in \mathrm{GL}_n(\mathbb{C})$ satisfying the relation $\prod_i g_i = 1$. The Riemann–Hilbert correspondence associates to a differential equation the tuple (g_i), where g_i is the monodromy of a full set of solutions at the singular point x_i.

In [**DW**] the authors investigated the following situation. Suppose that the set of points $\{x_1, \ldots, x_r\} \subset \mathbb{P}^1(\mathbb{C})$ and a local system $\mathcal{V}$ with singularities at the x_i depend on a parameter s which varies over the points of a complex manifold S. More precisely, we consider a relative divisor $D \subset \mathbb{P}^1_S$ of degree r such that for all $s \in S$ the fibre $D_s \subset \mathbb{P}^1(\mathbb{C})$ consists of r distinct points. Let $U := \mathbb{P}^1_S - D$ denote the complement

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and let $\mathcal{V}$ be a local system on U. We call $\mathcal{V}$ a *variation of local systems* over the base space S. The *parabolic cohomology* of the variation $\mathcal{V}$ is the local system on S

$$\mathcal{W} := R^1\pi_*(j_*\mathcal{V}),$$

where $j : U \hookrightarrow \mathbb{P}^1_S$ denotes the natural injection and $\pi : \mathbb{P}^1_S \to S$ the natural projection. The fibre of $\mathcal{W}$ at a point $s_0 \in S$ is the parabolic cohomology of the local system $\mathcal{V}_0$, the restriction of $\mathcal{V}$ to the fibre $U_0 = U \cap \pi^{-1}(s_0)$.

A special case of this construction is the *middle convolution functor* defined by Katz [**Kat97**]. Here $S = U_0$ and so this functor transforms one local system $\mathcal{V}_0$ on S into another one, $\mathcal{W}$. Katz shows that all rigid local systems on S arise from one-dimensional systems by successive application of middle convolution. This was further investigated by Dettweiler and Reiter [**DR03**]. Another special case are the generalized hypergeometric systems studied by Lauricella [**Lau93**], Terada [**Ter73**] and Deligne–Mostow [**DM86**]. Here S is the set of ordered tuples of pairwise distinct points on $\mathbb{P}^1(\mathbb{C})$ of the form $s = (0, 1, \infty, x_4, \ldots, x_r)$ and $\mathcal{V}$ is a one-dimensional system on $\mathbb{P}^1_S$ with regular singularities at the (moving) points $0, 1, \infty, x_4, \ldots, x_r$. In [**DW**] we gave another example where S is a 17-punctured Riemann sphere and the local system $\mathcal{V}$ has finite monodromy. The resulting local system $\mathcal{W}$ on S does not have finite monodromy and is highly non-rigid. Still, by the comparison theorem between singular and étale cohomology, $\mathcal{W}$ gives rise to ℓ-adic Galois representations, with interesting applications to the regular inverse Galois problem.

In all these examples, it is a significant fact that the monodromy of the local system $\mathcal{W}$ (i.e. the action of $\pi_1(S)$ on a fibre of $\mathcal{W}$) can be computed explicitly, i.e. one can write down matrices $g_1, \ldots, g_r \in \mathrm{GL}_n$ which are the images of certain generators $\alpha_1, \ldots, \alpha_r$ of $\pi_1(S)$. In the case of the middle convolution this was discovered by Dettweiler–Reiter [**DR00**] and Völklein [**Vö1**]. In [**DW**] it is extended to the more general situation sketched above. In all earlier papers, the computation of the monodromy is either not explicit (like in [**Kat97**]) or uses ad hoc methods. In contrast, the method presented in [**DW**] is very general and can easily be implemented on a computer.

It is one matter to compute the monodromy of $\mathcal{W}$ explicitly (i.e. to compute the matrices g_i) and another matter to determine its image (i.e. the group generated by the g_i). In many cases the image of monodromy is contained in a proper algebraic subgroup of GL_n, because $\mathcal{W}$ carries an invariant bilinear form induced from Poincaré duality. To compute the image of monodromy, it is often helpful to know this form explictly. After a review of the relevant results of [**DW**] in Section 1, we give a formula for the Poincaré duality pairing on $\mathcal{W}$ in Section 2. Finally, in Section 3 we illustrate our method in a very classical example: the Picard–Euler system.

1. Variation of parabolic cohomology revisited

1.1. Let X be a compact Riemann surface of genus 0 and $D \subset X$ a subset of cardinality $r \geq 3$. We set $U := X - D$. There exists a homeomorphism $\kappa : X \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$ between X and the Riemann sphere which maps the set D to the real line $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. Such a homeomorphism is called a *marking* of (X, D).

Having chosen a marking κ, we may assume that $X = \mathbb{P}^1(\mathbb{C})$ and $D \subset \mathbb{P}^1(\mathbb{R})$. Choose a base point $x_0 \in U$ lying in the upper half plane. Write $D = \{x_1, \ldots, x_r\}$ with $x_1 < x_2 < \cdots < x_r \leq \infty$. For $i = 1, \ldots, r-1$ we let γ_i denote the open interval $(x_i, x_{i+1}) \subset U \cap \mathbb{P}^1(\mathbb{R})$; for $i = r$ we set $\gamma_0 = \gamma_r := (x_r, x_1)$ (which may include ∞). For $i = 1, \ldots, r$, we let $\alpha_i \in \pi_1(U)$ be the element represented by a closed loop based at x_0 which first intersects γ_{i-1} and then γ_i. We obtain the following well known presentation

$$\pi_1(U, x_0) = \Big\langle\, \alpha_1, \ldots, \alpha_r \;\mid\; \prod_i \alpha_i = 1 \,\Big\rangle, \tag{1}$$

which only depends on the marking κ.

Let R be a (commutative) ring. A *local system of R-modules* on U is a locally constant sheaf $\mathcal{V}$ on U with values in the category of free R-modules of finite rank. Such a local system corresponds to a representation $\rho : \pi_1(U, x_0) \to \mathrm{GL}(V)$, where $V := \mathcal{V}_{x_0}$ is the stalk of $\mathcal{V}$ at x_0 (note that V is a free R-module of finite rank). For $i = 1, \ldots, r$, set $g_i := \rho(\alpha_i) \in \mathrm{GL}(V)$. Then we have

$$\prod_{i=1}^{r} g_i = 1,$$

and $\mathcal{V}$ can also be given by a tuple $\mathbf{g} = (g_1, \ldots, g_r) \in \mathrm{GL}(V)^r$ satisfying the above product-one-relation.

Convention 1.1. — Let α, β be two elements of $\pi_1(U, x_0)$, represented by closed path based at x_0. The composition $\alpha\beta$ is (the homotopy class of) the closed path obtained by first walking along α and then along β. Moreover, we let $\mathrm{GL}(V)$ act on V *from the right.*

1.2. Fix a local system of R-modules $\mathcal{V}$ on U as above. Let $j : U \hookrightarrow X$ denote the inclusion. The *parabolic cohomology* of $\mathcal{V}$ is defined as the sheaf cohomology of $j_*\mathcal{V}$, and is written as $H^n_p(U, \mathcal{V}) := H^n(X, j_*\mathcal{V})$. We have natural morphisms $H^n_c(U, \mathcal{V}) \to H^n_p(U, \mathcal{V})$ and $H^n_p(U, \mathcal{V}) \to H^n(U, \mathcal{V})$ (H_c denotes cohomology with compact support). Moreover, the group $H^n(U, \mathcal{V})$ is canonically isomorphic to the group cohomology $H^n(\pi_1(U, x_0), V)$ and $H^1_p(U, \mathcal{V})$ is the image of the cohomology with compact support in $H^1(U, \mathcal{V})$, see [**DW**, Prop. 1.1]. Thus, there is a natural inclusion

$$H^1_p(U, \mathcal{V}) \hookrightarrow H^1(\pi_1(U, x_0), V).$$

Let $\delta : \pi_1(U) \to V$ be a cocycle, i.e. we have $\delta(\alpha\beta) = \delta(\alpha) \cdot \rho(\beta) + \delta(\beta)$ (see Convention 1.1). Set $v_i := \delta(\alpha_i)$. It is clear that the tuple (v_i) is subject to the relation

$$v_1 \cdot g_2 \cdots g_r + v_2 \cdot g_3 \cdots g_r + \cdots + v_r = 0. \tag{2}$$

By definition, δ gives rise to an element in $H^1(\pi_1(U, x_0), V)$. We say that δ is a *parabolic* cocycle if the class of δ in $H^1(\pi_1(U), V)$ lies in $H^1_p(U, \mathcal{V})$. By [**DW**, Lemma 1.2], the cocycle δ is parabolic if and only if v_i lies in the image of $g_i - 1$, for all i. Thus, the assignment $\delta \mapsto (\delta(\alpha_1), \ldots, \delta(\alpha_r))$ yields an isomorphism

$$H^1_p(U, \mathcal{V}) \cong W_{\mathbf{g}} := H_{\mathbf{g}}/E_{\mathbf{g}}, \tag{3}$$

where

$$H_{\mathbf{g}} := \{\, (v_1, \ldots, v_r) \mid v_i \in \mathrm{Im}(g_i - 1), \text{relation (2) holds}\} \tag{4}$$

and

$$E_{\mathbf{g}} := \{\, (\, v \cdot (g_1 - 1), \ldots, v \cdot (g_r - 1)\,) \mid v \in V\}. \tag{5}$$

1.3. Let S be a connected complex manifold, and $r \geq 3$. An *r-configuration* over S consists of a smooth and proper morphism $\bar{\pi} : X \to S$ of complex manifolds together with a smooth relative divisor $D \subset X$ such that the following holds. For all $s \in S$ the fiber $X_s := \bar{\pi}^{-1}(s)$ is a compact Riemann surface of genus 0. Moreover, the natural map $D \to S$ is an unramified covering of degree r. Then for all $s \in S$ the divisor $D \cap X_s$ consists of r pairwise distinct points $x_1, \ldots, x_r \in X_s$.

Let us fix an r-configuration (X, D) over S. We set $U := X - D$ and denote by $j : U \hookrightarrow X$ the natural inclusion. Also, we write $\pi : U \to S$ for the natural projection. Choose a base point $s_0 \in S$ and set $X_0 := \bar{\pi}^{-1}(s_0)$ and $D_0 := X_0 \cap D$. Set $U_0 := X_0 - D_0 = \pi^{-1}(s_0)$ and choose a base point $x_0 \in U_0$. The projection $\pi : U \to S$ is a topological fibration and yields a short exact sequence

$$1 \longrightarrow \pi_1(U_0, x_0) \longrightarrow \pi_1(U, x_0) \longrightarrow \pi_1(S, s_0) \longrightarrow 1. \tag{6}$$

Let $\mathcal{V}_0$ be a local system of R-modules on U_0. A *variation* of $\mathcal{V}_0$ over S is a local system $\mathcal{V}$ of R-modules on U whose restriction to U_0 is identified with $\mathcal{V}_0$. The *parabolic cohomology* of a variation $\mathcal{V}$ is the higher direct image sheaf

$$\mathcal{W} := R^1\bar{\pi}_*(j_*\mathcal{V}).$$

By construction, $\mathcal{W}$ is a local system with fibre

$$W := H^1_p(U_0, \mathcal{V}_0).$$

(Since an r-configuration is locally trivial relative to S, it follows that the formation of $\mathcal{W}$ commutes with arbitrary basechange $S' \to S$.) Thus $\mathcal{W}$ corresponds to a representation $\eta : \pi_1(S, s_0) \to \mathrm{GL}(W)$. We call ρ the *monodromy representation* on the parabolic cohomology of $\mathcal{V}_0$ (with respect to the variation $\mathcal{V}$).

1.4. Under a mild assumption, the monodromy representation η has a very explicit description in terms of the *Artin braid group*. We first have to introduce some more notation. Define

$$\mathcal{O}_{r-1} := \{\, D' \subset \mathbb{C} \mid |D'| = r-1 \,\} = \{\, D \subset \mathbb{P}^1(\mathbb{C}) \mid |D| = r,\ \infty \in D \,\}.$$

The fundamental group $A_{r-1} := \pi_1(\mathcal{O}_{r-1}, D_0)$ is the *Artin braid group* on $r-1$ strands. Let $\beta_1, \dots, \beta_{r-2}$ be the standard generators, see e.g. [**DW**, § 2.2.] (The element β_i switches the position of the two points x_i and x_{i+1}; the point x_i walks through the lower half plane and x_{i+1} through the upper half plane.) The generators β_i satisfy the following well known relations:

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}, \qquad \beta_i\beta_j = \beta_j\beta_i \quad (\text{for } |i-j| > 1). \tag{7}$$

Let R be a commutative ring and V a free R-module of finite rank. Set

$$\mathcal{E}_r(V) := \{\, \mathbf{g} = (g_1, \dots, g_r) \mid g_i \in \mathrm{GL}(V),\ \prod_i g_i = 1 \,\}.$$

We define a right action of the Artin braid group A_{r-1} on the set $\mathcal{E}_r(V)$ by the following formula:

$$\mathbf{g}^{\beta_i} := (g_1, \dots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \dots, g_r). \tag{8}$$

One easily checks that this definition is compatible with the relations (7). For $\mathbf{g} \in \mathcal{E}_r(V)$, let $H_{\mathbf{g}}$ be as in (4). For all $\beta \in A_{r-1}$, we define an R-linear isomorphism

$$\Phi(\mathbf{g}, \beta) : H_{\mathbf{g}} \xrightarrow{\sim} H_{\mathbf{g}^\beta},$$

as follows. For the generators β_i we set

$$(v_1, \dots, v_r)^{\Phi(\mathbf{g},\beta_i)} := (v_1, \dots, v_{i+1}, \underbrace{v_{i+1}(1 - g_{i+1}^{-1} g_i g_{i+1}) + v_i g_{i+1}}_{(i+1)\text{th entry}}, \dots, v_r). \tag{9}$$

For an arbitrary word β in the generators β_i, we define $\Phi(\mathbf{g}, \beta)$ using (9) and the 'cocycle rule'

$$\Phi(\mathbf{g}, \beta) \cdot \Phi(\mathbf{g}^\beta, \beta') = \Phi(\mathbf{g}, \beta\beta'). \tag{10}$$

(Our convention is to let linear maps act from the right; therefore, the left hand side of (9) is the linear map obtained from first applying $\Phi(\mathbf{g}, \beta)$ and then $\Phi(\mathbf{g}^\beta, \beta')$.) It is easy to see that $\Phi(\mathbf{g}, \beta)$ is well defined and respects the submodule $E_{\mathbf{g}} \subset H_{\mathbf{g}}$ defined by (5). Let

$$\bar\Phi(\mathbf{g}, \beta) : W_{\mathbf{g}} \xrightarrow{\sim} W_{\mathbf{g}^\beta}$$

denote the induced map on the quotient $W_{\mathbf{g}} = H_{\mathbf{g}}/E_{\mathbf{g}}$.

Given $\mathbf{g} \in \mathcal{E}_r(V)$ and $h \in \mathrm{GL}(V)$, we define the isomorphism

$$\Psi(\mathbf{g}, h) : \left\{ \begin{array}{ccc} H_{\mathbf{g}^h} & \xrightarrow{\sim} & H_{\mathbf{g}} \\ (v_1, \dots, v_r) & \longmapsto & (v_1 \cdot h, \dots, v_r \cdot h). \end{array} \right. ,$$