

FAMILIES OF LINEAR DIFFERENTIAL EQUATIONS ON THE PROJECTIVE LINE

by

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Abstract. — The aim is to extend results of M.F. Singer on the variation of differential Galois groups. Let C be an algebraically closed field of characteristic 0. One considers certain families of connections of rank n on the projective line parametrized by schemes X over C . Let $G \subset \mathrm{GL}_n$ be an algebraic subgroup. It is shown that $X(= G)$, the set of closed points with differential Galois group G , is constructible for all families if and only if G satisfies a condition introduced by M.F. Singer. For the proof, techniques for handling families of vector bundles and connections are developed.

Résumé (Familles d'équations différentielles linéaires sur la droite projective)

Le but est de compléter des résultats de M.F. Singer concernant la variation des groupes de Galois différentiels. Soit C un corps algébriquement clos, de caractéristique 0. On considère des familles de connections de rang n sur la droite projective, paramétrisées par des schémas X sur C . Soit $G \subset \mathrm{GL}_n$ un sous-groupe algébrique. On montre que $X(= G)$, l'ensemble des points fermés de X avec G comme groupe de Galois différentiel, est constructible pour toute famille si et seulement si le groupe G satisfait une condition introduite par M.F. Singer. Pour la démonstration, des techniques concernant des familles de fibrés vectoriels et des connections sont développées.

1. Introduction

C is an algebraically closed field of characteristic 0 and X denotes a scheme of finite type over C . We fix a vector space V of dimension n over C and an algebraic subgroup G of $\mathrm{GL}(V)$. We will define *families of linear differential equations* on the projective line C , parametrized by X . These families are of a more general nature than the moduli spaces, defined in [Ber02]. For each closed point x of X (i.e., $x \in X(C)$), the differential equation corresponding to x has a differential Galois group, denoted by $\mathrm{Gal}(x)$. It is shown that the condition “ $\mathrm{Gal}(x) \subset G$ ” for closed points x of X defines a closed subset of X . This generalizes Theorem 4.2 of [Ber02], where this statement is proved for moduli spaces.

2000 Mathematics Subject Classification. — 13N, 14D, 20G, 53G.

Key words and phrases. — Connections, moduli, differential Galois group.

The aim is to show that the set of closed points $x \in X$ for which the differential Galois group $\text{Gal}(x)$ of the corresponding equation is equal to G is a *constructible* subset of X , i.e., of the form $\cup_{i=1}^n (O_i \cap F_i)$ for open sets O_i and closed sets F_i . This statement (and the earlier one) has to be made more precise by providing a suitable definition of “family of differential equations” and a meaning for the expression $\text{Gal}(x) \subset G$. Moreover, a condition on the group G is essential.

In his paper [Sin93], M.F. Singer defines a set of differential operators, by giving some local data. He proves that under a certain condition on G , the subset of the differential equations with Galois group equal to (a conjugate of) G is constructible. This condition on G will be called the *Singer condition*. We consider the same problem, in our context of families of differential equations parametrized by a scheme X . We will construct for any group G that does not satisfy the “Singer condition” an example of a moduli family $\mathbb{M}$ such that $\{x \in \mathbb{M} \mid \text{Gal}(x) = G\}$ is not constructible. Finally, from these constructions one deduces an alternative description of the Singer condition.

2. The Singer condition

Let G be a linear algebraic group over C . First we will recall the Singer condition on G , as given in [Sin93]. A character χ of G is a morphism of algebraic groups $\chi : G \rightarrow \mathbb{G}_m$, where $\mathbb{G}_m$ stands for the multiplicative group C^* . The set $X(G)$ of all characters is a finitely generated abelian group. Let $\ker X(G)$ denote the intersection of the kernels of all $\chi \in X(G)$. This intersection is a characteristic (closed) subgroup of G . As usual, G^o denotes the connected component of the identity of G . The group $\ker X(G^o)$ is a normal, closed subgroup of G^o and of G . Let $\chi_1, \dots, \chi_s$ generate $X(G^o)$. Then $\ker X(G^o)$ is equal to the intersection of the kernels of $\chi_1, \dots, \chi_s$. In other words $\ker X(G^o)$ is the kernel of the morphism $G^o \rightarrow \mathbb{G}_m^s$, given by $g \mapsto (\chi_1(g), \dots, \chi_s(g))$. The image is a connected subgroup of $\mathbb{G}_m^s$ and therefore a torus T . Hence $G^o/\ker X(G^o)$ is isomorphic to T . Moreover, by definition, T is the largest torus factor group of G^o . One considers the exact sequence:

$$1 \longrightarrow G^o/\ker X(G^o) \longrightarrow G/\ker X(G^o) \longrightarrow G/G^o \longrightarrow 1.$$

Since $G^o/\ker X(G^o)$ is abelian, this sequence induces an action of G/G^o on $G^o/\ker X(G^o)$ by conjugation.

Definition 2.1. — A linear algebraic group G satisfies the *Singer Condition* if the action of G/G^o on $G^o/\ker X(G^o)$ is trivial.

The Singer condition can be stated somewhat simpler, using $U(G) \subset G$, the subgroup generated by all unipotent elements in G .

Lemma 2.2. — $U(G) = U(G^o)$ is equal to $\ker X(G^o)$ and the Singer condition is equivalent to “ $G^o/U(G)$ lies in the center of $G/U(G)$ ”.

Proof. — Fix an embedding $G \subset \mathrm{GL}(V)$, where V is a finite dimensional vector space over C . First we prove that $U(G)$ is a closed connected normal subgroup of G . Let $I + B$, $B \neq 0$ be a unipotent element of G . Then $I + B = e^D$, for some nilpotent element $D = \sum (-1)^{i-1} \frac{B^i}{i} \in \mathrm{End}(V)$. The Zariski closure $\overline{\{(I + B)^n | n \in \mathbb{Z}\}}$ of the group generated by $I + B$ lies in G and is equal to the group $\{e^{tD} | t \in C\}$, which is isomorphic to the additive group $\mathbb{G}_a$ over C . Hence $U(G)$ is generated by these connected subgroups of G and by Proposition 2.2.6 of [Spr98] the group $U(G)$ is closed and connected. Further $U(G)$ is a normal subgroup and even a characteristic subgroup, since the set of unipotent elements of G is stable under any automorphism of G . The connectedness of $U(G)$ implies $G^\circ \supset U(G) = U(G^\circ)$.

Now we will show that $G^\circ/U(G^\circ)$ is a torus. Since the *unipotent radical* $R_u(G^\circ)$ lies in $U(G^\circ)$, we may divide G° by $R_u(G^\circ)$ and assume that G° to be *reductive*. Then by [Spr98][corollary 8.1.6] we have $G^\circ = R(G^\circ) \cdot (G^\circ, G^\circ)$, where $R(G^\circ)$ is the radical of G° , and where (G°, G°) is the *commutator subgroup* of G° . The latter group is a semi-simple subgroup, according to the same corollary. By [Spr98][theorem 8.1.5] we get that (G°, G°) is generated by unipotent elements, so $(G^\circ, G^\circ) \subset U(G^\circ)$. Since $R(G^\circ)$ is a torus, its image $G^\circ/U(G^\circ)$ is a torus, too. This proves $U(G^\circ) \supset \ker X(G^\circ)$. The other inclusion follows from the observation that every unipotent element lies in the kernel of every character.

Finally, the triviality of the action of G/G° on $G^\circ/U(G^\circ)$ is clearly equivalent to $G^\circ/U(G^\circ)$ lies in the center of $G/U(G^\circ)$. $\square$

Remarks 2.3

(1) Let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup. *For the moment we admit the following items* (see 3.4, 3.5 (2), 4.1 and 4.2):

- The definition of a family of differential equations, parametrized by X .
- The meaning of $\mathrm{Gal}(x) \subset G$ for $x \in X(C)$.
- $\{x \in X(C) \mid \mathrm{Gal}(x) \subset G\}$ is closed.
- $\{x \in X(C) \mid \mathrm{Gal}(x) \subset hGh^{-1} \text{ for some } h \in \mathrm{GL}(V)\}$ is constructible.

Consider the following finiteness condition for the group G : (*) G has finitely many proper closed subgroups $H_1, \dots, H_s$, such that every proper closed subgroup is contained in a conjugate of one of the H_i . One easily deduces: *If G satisfies (*), then $\{x \in X(C) \mid \mathrm{Gal}(x) = G\}$ is constructible.*

(2) *If G satisfies (*), then $G/U(G)$ is a finite group and in particular G satisfies the Singer condition.* Indeed, (*) also holds for $G/U(G)$. If $T := G^\circ/U(G) \neq \{1\}$, then one can produce infinitely many proper normal subgroups of $G/U(G)$. Namely, for any integer $m > 1$ the subgroup $T[m]$, consisting of the m -torsion elements of T , is a normal subgroup. One concludes that $G/U(G)$ is finite.

(3) Consider $G := \mathrm{SL}_2(C)$. The classification of the proper closed subgroups H of G states that H is either contained in a Borel subgroup or in a conjugate of the

infinite dihedral group $D_\infty^{\text{SL}_2}$ or is conjugated to one of the special finite groups: the tetrahedral group, the octahedral group, the icosahedral group. Thus G satisfies $(*)$ and moreover, $G/U(G) = \{1\}$.

(4) The infinite dihedral group $G = D_\infty^{\text{SL}_2}$ has the properties: $G^o = \mathbb{G}_m$, $U(G^o) = 1$ and G/G^o acts non-trivially on G^o . Thus G does not satisfy the Singer condition. For this group one can produce moduli spaces $\mathbb{M}$ such that $\{x \in \mathbb{M}(C) \mid \text{Gal}(x) = G\}$ is not constructible (see example 2.6).

(5) For the following two examples, namely moduli spaces and the groups $\mathbb{G}_a^3$ and $\mathbb{G}_m^n$, the Singer condition is valid, but $(*)$ does not hold. We will show explicitly that these groups define constructible subsets.

Example 2.4 (A moduli space with differential Galois groups in $\mathbb{G}_a^3$)

Moduli spaces of the type considered here are defined in [Ber02]. V is a 4-dimensional vector space over C with basis $e_1, \dots, e_4$. The element $N \in \text{End}(V)$ is given by $N(e_i) = 0$ for $i = 1, 2, 3$ and $N(e_4) = e_1$. The data for the moduli problem are:

- Three distinct singular points $a_1, a_2, a_3 \in C^*$. The point ∞ is allowed to have a, non prescribed, regular singularity.
- For each singular point a_i , the differential operator $\frac{d}{d(z-a_i)} + \frac{N}{z-a_i}$.

Some calculations lead to an identification $\text{GL}(4, C) \times \text{GL}(4, C) \rightarrow \mathbb{M}$, where $\mathbb{M}$ is the moduli space of the problem. Let $m := (\phi_2, \phi_3)$ denote a closed point of the first space, then the corresponding universal differential operator is

$$\frac{d}{dz} + \frac{N}{z-a_1} + \frac{\phi_2 N \phi_2^{-1}}{z-a_2} + \frac{\phi_3 N \phi_3^{-1}}{z-a_3}.$$

Let $G := \mathbb{G}_a^3$ the subgroup of $\text{GL}(V)$ consisting of the maps of the form $I+B$, $Be_1 = 0$ and $Be_i \in Ce_1$ for $i = 2, 3, 4$. The condition $\text{Gal}(m) \subset \mathbb{G}_a^3$ can be seen to be equivalent to $\phi_2(e_1), \phi_3(e_1) \in Ce_1$. This describes the set $\{m \in \mathbb{M} \mid \text{Gal}(m) \subset G\}$ completely. The above differential operator evaluated at a point of $\{m \in \mathbb{M} \mid \text{Gal}(m) \subset G\}$ has the form

$$\frac{d}{dz} + \begin{pmatrix} 0 & h_1 & h_2 & h_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ where}$$

$$(h_1, h_2, h_3) = \frac{1}{z-a_1}(0, 0, 1) + \frac{1}{z-a_2}(f_1, f_2, f_3) + \frac{1}{z-a_3}(g_1, g_2, g_3).$$

Moreover, f_1, f_2, f_3 are polynomials of degree ≤ 2 in the entries of ϕ_2 and the g_1, g_2, g_3 are polynomials of degree ≤ 2 in the entries of ϕ_3 .

Now G has infinitely many (non-conjugated) maximal proper closed subgroups and there is no obvious reason why $\{m \in \mathbb{M} \mid \text{Gal}(m) = G\}$ should be constructible. We continue the calculation. The differential Galois group $\text{Gal}(m)$, with m such that

$\text{Gal}(m) \subset G$, is in fact the differential Galois group for the three inhomogeneous equations $y'_i = h_i$, $i = 1, 2, 3$ over $C(z)$. Thus $\text{Gal}(m)$ is a proper subgroup of G if and only if there is a non trivial linear combination $c_1 h_1 + c_2 h_2 + c_3 h_3$ with $c_1, c_2, c_3 \in C$ such that $y' = c_1 h_1 + c_2 h_2 + c_3 h_3$ has a solution in $C(z)$. Now y exists if and only if $c_1 h_1 + c_2 h_2 + c_3 h_3$ has residue 0 at the points a_1, a_2, a_3 . The existence of such a linear combination translates into a linear dependence and the explicit equation $f_1(a_2)g_2(a_3) - f_2(a_2)g_1(a_3) = 0$. This defines a closed subset of $\{m \in \mathbb{M} \mid \text{Gal}(m) \subset G\}$ and so $\{m \in \mathbb{M} \mid \text{Gal}(m) = G\}$ is constructible. We note that every linear subspace of $\mathbb{G}_a^3 \cong C^3$, which contains $(0, 0, 1)$, occurs as differential Galois group.

Example 2.5 (A moduli space with differential Galois groups in $\mathbb{G}_m^n$)

The data for the moduli problem are:

- A vector space V of dimension n over C and basis $e_1, \dots, e_n$.
- Singular points $a_1, \dots, a_s$, different from 0 and ∞ . We allow ∞ to have a non-prescribed regular singularity.
- Local differential operators $\frac{d}{d(z-a_i)} + \frac{M_i}{z-a_i}$, where $e_1, \dots, e_n$ are eigenvectors for all $M_i \in \text{End}(V)$.

The moduli space $\mathbb{M}$ can be identified with $\text{GL}(V)^{s-1}$. At a closed point $m = (\phi_2, \dots, \phi_s) \in \text{GL}(V)^{s-1}$ the universal differential operator reads

$$\frac{d}{dz} + \sum_{i=1}^s \frac{\phi_i M_i \phi_i^{-1}}{z - a_i},$$

where $\phi_1 = I$. The group $\mathbb{G}_m^n \cong G \subset \text{GL}(V)$ consists of the maps for which each e_i is an eigenvector. Above the closed subset $\{m \in \mathbb{M} \mid \text{Gal}(m) \subset G\}$ the differential operator has the form

$$L := \frac{d}{dz} + \sum_{i=1}^s \frac{N_i}{z - a_i},$$

with $N_1 = M_1$ and each N_i is a diagonal matrix w.r.t. the basis $e_1, \dots, e_n$ and having the same eigenvalues as M_i . The space $\{m \in \mathbb{M} \mid \text{Gal}(m) \subset G\}$ has positive dimension and is rather large if there is at least one M_i with $i > 1$ having an eigenvalue with multiplicity > 1 . However the number of differential operators L is finite! Thus only a finite number of algebraic subgroups of $G \cong \mathbb{G}_m^n$ occur as differential Galois group $\text{Gal}(m)$. One concludes that for every algebraic subgroup $H \subset G$, the set $\{m \in \mathbb{M} \mid \text{Gal}(m) = H\}$ is constructible.

This example is the general pattern for “families” with differential Galois groups contained in some torus T . Again, there are only finitely many distinct differential operators L possible and therefore only finitely many possibilities for the differential Galois group. This implies that for every algebraic subgroup $H \subset T$ the set of the points with differential Galois group equal to H is constructible.