

## SOME LIMITING SITUATIONS FOR SEMILINEAR ELLIPTIC EQUATIONS

by

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**Abstract.** — The objective of this mini-course is to take a look at a standard semilinear partial differential equation  $-\Delta u = \lambda f(u)$  on which we show the use of some basic tools in the study of elliptic equation. We will mention the maximum principle, barrier method, blow-up analysis, regularity and boot-strap argument, stability, localization and quantification of singularities, Pohozaev identities, moving plane method, etc.

**Résumé (Quelques situations limites pour les équations semi-linéaires elliptiques)**

L'objectif de ce mini-cours est de jeter un coup d'œil sur une équation aux dérivées partielles standard  $-\Delta u = \lambda f(u)$ , avec laquelle nous allons montrer quelques outils de base dans l'étude des équations elliptiques. Nous mentionnerons le principe du maximum, la méthode de barrière, l'analyse de blow-up, la régularité, l'argument de boot-strap, la stabilité, la localisation et quantification de singularités, les identités de Pohozaev, la méthode du plan mobile, etc.

### 1. Introduction

We consider the following semilinear partial differential equation:

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain and  $f$  is a smooth positive, nondecreasing and convex function over  $\mathbb{R}_+$ . For getting a positive solution  $u$ , necessarily  $\lambda$  is positive.

The convexity of  $f$  implies that

- $\lim_{t \rightarrow \infty} f(t)/t = a \in \mathbb{R}_+ \cup \{\infty\}$  exists.
- If  $a \in \mathbb{R}_+$ , then  $\lim_{t \rightarrow \infty} f(t) - at = l \in \mathbb{R} \cup \{-\infty\}$  exists.

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Since the case  $a = 0$  is trivial ( $f \equiv \text{constant}$ ), we will suppose that  $a > 0$ . Thus we can divide the study of problem  $(P_\lambda)$  into two different situations: the quasilinear case when  $a \in (0, \infty)$  and superlinear case when  $a = \infty$ . We will see that the first case is rather well understood, while many questions are remained open for the second one.

In the following,  $\|\cdot\|_p$  denotes the standard  $L^p$  norm for  $1 \leq p \leq \infty$ .  $W^{1,p}(\Omega)$  is the Sobolev space of functions  $f$  such that  $f$  and  $\nabla f \in L^p(\Omega)$ . When  $p = 2$ , we use for simplicity  $H^1(\Omega)$  to denote  $W^{1,2}(\Omega)$ ,  $H_0^1(\Omega)$  is the space of functions  $f \in H^1(\Omega)$  verifying  $f = 0$  on  $\partial\Omega$ . The symbol  $C$  means always a positive constant independent of  $\lambda$ .

## 2. Quasilinear situation

We begin with the quasilinear case where  $a \in (0, \infty)$ . Many results presented here are obtained by Mironescu & Rădulescu in [27].

**2.1. Minimal solution and stability.** — Since  $f(u) \leq au + f(0)$  in this case, then if  $u \in L^1(\Omega)$  is a weak solution of  $(P_\lambda)$  in the sense of distribution, we get easily that  $u$  is always a classical solution by standard boot-strap argument.

**Lemma 2.1.** — *For  $\lambda > 0$ , if  $(P_\lambda)$  is resolvable, then a minimal solution  $u_\lambda$  exists in the sense that any solution  $v$  of  $(P_\lambda)$  verifies  $v \geq u_\lambda$  in  $\Omega$ . Moreover,  $(P_{\lambda'})$  is resolvable for any  $\lambda' \in (0, \lambda)$ .*

*Proof.* — We will use the barrier method. Remark that for  $\lambda > 0$ ,  $w_0 \equiv 0$  is a sub solution of  $(P_\lambda)$  since  $f(0) > 0$ . Now we define for any  $n \in \mathbb{N}$ ,

$$(1) \quad -\Delta w_{n+1} = \lambda f(w_n) \text{ in } \Omega, \quad w_{n+1} = 0 \text{ on } \partial\Omega.$$

Using maximum principle,  $w_1 > w_0 \equiv 0$  in  $\Omega$ . On the other hand, let  $v$  be any solution of  $(P_\lambda)$ , by monotonicity of  $f$ , we obtain

$$-\Delta(w_1 - v) = \lambda[f(0) - f(v)] \leq 0 \text{ in } \Omega, \quad w_1 - v = 0 \text{ on } \partial\Omega.$$

Thus  $w_1 \leq v$  in  $\Omega$ . We can prove by induction that the sequence  $\{w_n\}$  verifies  $w_n \leq w_{n+1} \leq v$  in  $\Omega$  for any  $n$ , so  $u_\lambda = \lim_{n \rightarrow \infty} w_n$  is well defined, and  $u_\lambda$  is a solution of  $(P_\lambda)$  by passing to the limit in (1). Moreover,  $u_\lambda \leq v$ . Notice that the definition of  $u_\lambda$  is independent of the choice of  $v$ , it is the minimal solution claimed.

If  $(P_\lambda)$  has a solution  $u$ , it is a super solution for  $(P_{\lambda'})$  when  $0 < \lambda' < \lambda$ . As  $w_0 \equiv 0$  is always a sub solution, the barrier method will solve as above  $(P_{\lambda'})$ .  $\square$

Let  $\lambda_1$  be the first eigenvalue of  $-\Delta$  on  $\Omega$  with the Dirichlet boundary condition, we define  $\varphi_0$  to be the first eigenfunction such that  $\varphi_0 > 0$  in  $\Omega$  and  $\|\varphi_0\|_2 = 1$ .

**Lemma 2.2.** — *If we denote  $r_0 = \inf_{t>0} f(t)/t$ , then  $(P_\lambda)$  has no solution for  $\lambda > \lambda_1/r_0$ . On the other hand,  $(P_\lambda)$  is resolvable for  $\lambda > 0$  small enough.*

*Proof.* — Let  $\xi \in H_0^1(\Omega) \cap L^\infty(\Omega)$  be the solution of  $-\Delta\xi = 1$  in  $\Omega$ . It is easy to see that  $\xi$  is a super solution of  $(P_\lambda)$  for  $\lambda \leq f(\|\xi\|_\infty)^{-1}$ . Applying the barrier method, we get a solution of  $(P_\lambda)$  for such  $\lambda$ .

Now we suppose that  $u$  is a solution of  $(P_\lambda)$  for some  $\lambda > 0$ , using  $\varphi_0$  as test function and integrating by parts, we get

$$\lambda_1 \int_{\Omega} \varphi_0 u dx = - \int_{\Omega} u \Delta \varphi_0 dx = - \int_{\Omega} \varphi_0 \Delta u dx = \lambda \int_{\Omega} f(u) \varphi_0 dx.$$

As  $f(u) \geq r_0 u$  in  $\Omega$ , we have then

$$(\lambda_1 - \lambda r_0) \int_{\Omega} \varphi_0 u dx \geq 0.$$

Recalling that  $\varphi_0$  and  $u$  are positive in  $\Omega$ , the lemma is proved.  $\square$

Combining these two lemmas, we can claim

**Theorem 2.3.** — *There exists a critical value  $\lambda^* \in (0, \infty)$  for the parameter  $\lambda$ , such that for any  $\lambda > \lambda^*$ , no solution exists for the problem  $(P_\lambda)$  while for any  $\lambda \in (0, \lambda^*)$ , a unique minimal solution  $u_\lambda$  exists for  $(P_\lambda)$ . Furthermore the mapping  $\lambda \mapsto u_\lambda$  is increasing with  $\lambda$ .*

It is natural to ask if we can determine the exact value of  $\lambda^*$  and what happens when  $\lambda = \lambda^*$ . Before considering these two questions, we show another characterization of the minimal solution  $u_\lambda$ , its stability. A solution  $u$  of  $(P_\lambda)$  is called stable if and only if the linearized operator associated to the equation,  $-\Delta - \lambda f'(u)$  is nonnegative. More precisely,

$$(2) \quad \lambda \int_{\Omega} f'(u) \varphi^2 dx \leq \int_{\Omega} |\nabla \varphi|^2 dx, \quad \text{for any } \varphi \in H_0^1(\Omega).$$

**Theorem 2.4.** — *Let  $\lambda \in (0, \lambda^*)$ , the minimal solution  $u_\lambda$  is the unique stable solution of  $(P_\lambda)$ .*

*Proof.* — First we prove that  $u_\lambda$  is stable. If it is not true, the first eigenvalue  $\eta_1$  of  $-\Delta - \lambda f'(u_\lambda)$  is negative, then there exists an eigenfunction  $\psi \in H_0^1(\Omega)$  such that

$$-\Delta \psi - \lambda f'(u_\lambda) \psi = \eta_1 \psi \quad \text{in } \Omega, \quad \psi > 0 \quad \text{in } \Omega.$$

Consider  $u^\varepsilon = u_\lambda - \varepsilon \psi$ , a direct calculation gives

$$-\Delta u^\varepsilon - \lambda f(u^\varepsilon) = -\eta_1 \varepsilon \psi - \lambda [f(u_\lambda - \varepsilon \psi) - f(u_\lambda) + \varepsilon f'(u_\lambda) \psi] = \varepsilon \psi [-\eta_1 + o_\varepsilon(1)].$$

Since  $\eta_1 < 0$ , then  $-\Delta u^\varepsilon - \lambda f(u^\varepsilon) \geq 0$  in  $\Omega$  for  $\varepsilon > 0$  small enough. Otherwise, using Hopf's lemma, we know that  $u_\lambda \geq C\psi$  in  $\Omega$  for some  $C > 0$ . Thus  $u^\varepsilon \geq 0$  is a super solution of  $(P_\lambda)$  for  $\varepsilon > 0$  small enough. As before, we can get a solution  $u$  such that  $u \leq u^\varepsilon$  in  $\Omega$ , which contradicts the minimality of  $u_\lambda$ . So  $\eta_1 \geq 0$ .

Now we prove that  $(P_\lambda)$  has at most one stable solution. Suppose the contrary, there exists another stable solution  $v \neq u_\lambda$ . Define  $\varphi = v - u_\lambda$ , we get

$$\lambda \int_{\Omega} f'(v) \varphi^2 dx \leq \int_{\Omega} |\nabla \varphi|^2 dx = - \int_{\Omega} \varphi \Delta \varphi dx = \lambda \int_{\Omega} [f(v) - f(u_\lambda)] \varphi dx,$$

so

$$\int_{\Omega} [f(v) - f(u_{\lambda}) - f'(v)(v - u_{\lambda})] \varphi dx \geq 0.$$

By maximum principle, we know that  $\varphi > 0$  in  $\Omega$ . The convexity of  $f$  yields that the term in the bracket is non positive, so the only possibility is  $f(v) - f(u_{\lambda}) - f'(v)(v - u_{\lambda}) \equiv 0$  in  $\Omega$ , which means  $f$  is affine over  $[u_{\lambda}(x), v(x)]$  for any  $x \in \Omega$ . Thus  $f(x) = \bar{a}x + b$  in  $[0, \max_{\Omega} v]$  and we get two solutions  $u$  and  $v$  of  $-\Delta w = \bar{a}w + b$ . This implies that

$$0 = \int_{\Omega} u_{\lambda} \Delta v - v \Delta u_{\lambda} dx = b \int_{\Omega} (v - u_{\lambda}) dx = b \int_{\Omega} \varphi dx,$$

which is impossible since  $b = f(0) > 0$  and  $\varphi$  is positive in  $\Omega$ . So we are done.  $\square$

An immediate consequence of Theorem 2.4 is

**Proposition 2.5.** — *For any  $\lambda \in (0, \lambda_1/a)$ ,  $(P_{\lambda})$  has one and unique solution  $u_{\lambda}$ .*

*Proof.* — Remark first  $a = \sup_{\mathbb{R}_+} f'(t)$  by convexity of  $f$ . Thanks to the definition of  $\lambda_1$ , it is clear that each solution is stable if  $\lambda \in (0, \lambda_1/a)$ , so we get the uniqueness by that for stable solution. For the existence, we can consider the minimization problem  $\min_{H_0^1(\Omega)} J(u)$  where

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} F(u) dx$$

with

$$F(u) = \int_0^{u^+} f(s) ds, \quad u^+ = \max(u, 0).$$

If  $\lambda \in (0, \lambda_1/a)$ , there exist  $\varepsilon, A > 0$  depending on  $\lambda$  such that  $2\lambda F(t) \leq (\lambda_1 - \varepsilon)t^2 + A$  over  $\mathbb{R}$ . Thus  $J(u)$  is coercive, bounded from below and weakly lower semi-continuous in  $H_0^1(\Omega)$ , the infimum of  $J$  is reached then by a function  $u \in H_0^1(\Omega)$ , so also by  $u^+ \in H_0^1(\Omega)$  since  $J(u^+) \leq J(u)$ . This critical point  $u \geq 0$  of  $J$  gives a solution of  $(P_{\lambda})$ .  $\square$

**2.2. Estimate of  $\lambda^*$ .** — By Proposition 2.5, we know that  $\lambda^* \geq \lambda_1/a$ . The following result in [27] gives us more precise information for  $\lambda^*$ .

**Theorem 2.6.** — *We have three equivalent assertions:*

- (i)  $\lambda^* = \lambda_1/a$ .
- (ii) No solution exists for  $(P_{\lambda^*})$ .
- (iii)  $\lim_{\lambda \rightarrow \lambda^*} u_{\lambda} = \infty$  u.c. in  $\Omega$ . (u.c. means “uniformly on each compact set”)

*Proof.* — (i) implies (ii). If  $(P_{\lambda^*})$  has a solution  $u$ , then  $u_{\lambda} \leq u$  in  $\Omega$  for any  $\lambda \in (0, \lambda^*)$ , using the monotonicity of  $u_{\lambda}$ ,  $u^* = \lim_{\lambda \rightarrow \lambda^*} u_{\lambda}$  is well defined and  $u^*$  is clearly a stable solution of  $(P_{\lambda^*})$  by limit. Consider the operator  $G(u, \lambda) = -\Delta u - \lambda f(u)$ , if the first eigenvalue  $\eta_1$  of  $-\Delta - \lambda^* f'(u^*)$  is positive, then we can apply the Implicit Function Theorem to get a solution curve in a neighborhood of  $\lambda^*$ , but this contradicts the definition of  $\lambda^*$ , so  $\eta_1 = 0$ . Thus, there exists  $\psi \in H_0^1(\Omega)$  satisfying

$$(3) \quad -\Delta \psi - \lambda^* f'(u^*) \psi = 0 \quad \text{and} \quad \psi > 0 \quad \text{in } \Omega.$$

Using  $\varphi_0$  as test function and integrating by parts, we get

$$\int_{\Omega} [\lambda_1 - \lambda^* f'(u^*)] \psi \varphi_0 dx = 0.$$

As  $\lambda_1 - \lambda^* f'(u^*) \geq 0$ , we get  $f'(u^*) \equiv a$  in  $\Omega$  so that  $f(t) = at + b$  in  $[0, \max_{\Omega} u^*]$ . But  $b > 0$  deduces that no positive solution in  $H_0^1(\Omega)$  can exist for the equation  $-\Delta u = \lambda_1 u + b\lambda_1/a$  (we can use again  $\varphi_0$ ), so the hypothesis is not true.

(ii) implies (iii). Here we mention a result of Hörmander (see [22]) as follows: *For a sequence of nonnegative super-harmonic functions  $\{v_n\}$  in  $\Omega$ , either  $v_n$  converges u.c. to  $\infty$ ; or there exists a subsequence which converges in  $L_{loc}^1(\Omega)$ .* We need just to prove that the second case cannot occur for  $u_{\lambda}$ . Suppose the contrary, there exist  $u_k = u_{\lambda_k}$  which converges in  $L_{loc}^1(\Omega)$  to  $u^*$  with  $\lambda_k \rightarrow \lambda^*$ . We claim that  $\|u_k\|_2 \leq C$ . If it is false, we define  $u_k = l_k w_k$  with  $\|w_k\|_2 = 1$  and  $\lim_{k \rightarrow \infty} l_k = \infty$  (up to subsequence). Since  $f(t) \leq at + f(0)$ ,

$$-\Delta w_k = \frac{\lambda_k f(u_k)}{l_k} \leq a\lambda_k w_k + \frac{\lambda_k f(0)}{l_k} \leq a\lambda_k w_k + C \quad \text{in } \Omega,$$

it is easy to see that  $w_k$  is bounded in  $H_0^1(\Omega)$ , so that up to a subsequence,  $w_k$  converges weakly in  $H_0^1$  and strongly in  $L^2$  to some  $w \in H_0^1$ . Meanwhile,  $-\Delta w_k$  tends to zero in  $L_{loc}^1(\Omega)$  since  $f(u_k) \leq au_k + b$  and  $l_k$  tends to  $\infty$ , this implies  $-\Delta w = 0$  in  $\Omega$ . Hence  $w \equiv 0$ , which is impossible because  $\|w\|_2 = \lim_{k \rightarrow \infty} \|w_k\|_2 = 1$ . So  $\{u_k\}$  is bounded in  $L^2(\Omega)$ , hence in  $H_0^1(\Omega)$  by equation. We prove readily that  $u^*$  is a solution of  $(P_{\lambda^*})$  which contradicts (ii).

(iii) implies (ii). Any solution  $u$  of  $(P_{\lambda^*})$  should satisfy  $u \geq u_{\lambda}$ ,  $\forall \lambda < \lambda^*$ .

(ii)  $\oplus$  (iii) implies (i). Clearly  $\lim_{\lambda \rightarrow \lambda^*} \|u_{\lambda}\|_2 = \infty$ . Take  $u_{\lambda} = l_{\lambda} w_{\lambda}$  with  $\|w_{\lambda}\|_2 = 1$ , then we have a subsequence  $w_k$  which converges weakly in  $H_0^1$ , strongly in  $L^2$  and almost everywhere to  $w \geq 0$ . Moreover, in the sense of distribution,

$$-\Delta w = - \lim_{D'(\Omega)} \Delta w_k = \lim \frac{\lambda_k f(l_k w_k)}{l_k} = \lambda^* a w \geq \lambda_1 w \quad \text{a.e.}$$

Taking again  $\varphi_0$  as test function, we see that the last inequality must be an equality, so  $\lambda^* = \lambda_1/a$ .  $\square$

Remark that when  $f(t) \geq at$  in  $\mathbb{R}_+$ , we cannot get a solution for  $\lambda = \lambda_1/a$  since  $f(t) > at$  in a neighborhood of 0 (using always  $\varphi_0$  as test function), we obtain an important consequence of Theorem 2.6 and Proposition 2.5.

**Corollary 2.7.** — *If we have  $\lim_{t \rightarrow \infty} f(t) - at = l \geq 0$ , then  $\lambda^* = \lambda_1/a$ , and a unique solution  $u_{\lambda}$  exists for  $(P_{\lambda})$  for  $\lambda \in (0, \lambda^*)$  while no solution exists for  $\lambda \geq \lambda^*$ .*

Moreover, the following result is established in [27].

**Proposition 2.8.** — *If  $\lim_{t \rightarrow \infty} f(t) - at = l < 0$ , then  $\lambda_1/a < \lambda^* < \lambda_1/r_0$ . A unique solution  $u^* = \lim_{\lambda \rightarrow \lambda^*} u_{\lambda}$  exists for  $(P_{\lambda^*})$ . Furthermore, for any  $\lambda \in (\lambda_1/a, \lambda^*)$ , we have a second solution  $v_{\lambda}$  for  $(P_{\lambda})$ , such that  $v_{\lambda}$  tends u.c. to  $\infty$  in  $\Omega$  when  $\lambda \downarrow \lambda_1/a$ .*