CLASSICAL, EXCEPTIONAL, AND EXOTIC HOLONOMIES : A STATUS REPORT

Robert L. BRYANT

Department of Mathematics
Duke University
PO Box 90320
Durham, NC 27708–0320, USA
bryant@math.duke.edu

Abstract. I report on the status of the problem of determining the groups that can occur as the irreducible holonomy of a torsion-free affine connection on some manifold.

Résumé. Il s'agit d'un rapport sur le problème de la détermination des groupes qui peuvent être les groupes d'holonomie de connexions affines sans torsion.

M.S.C. Subject Classification Index (1991) : 53B05 53C10; 53C15 58A15.

Acknowledgements. This manuscript was begun during the Round Table at Luminy in 1992. The support of the National Science Foundation in the form of Grant DMS-9205222 is gratefully acknowledged.

TABLE OF CONTENTS

INTRODUCTION

0.1. Overview. The goal of this report is to present, in a unified way, what is known about the problem of prescribed holonomy of torsion-free affine connections smooth manifolds.

In §1, I give the fundamental definitions and develop the algebra needed to formulate Berger's criteria which a subgroup of $\mathrm{GL}(T_xM)$ must satisfy if it is to be the holonomy of a torsion-free affine connection on M which is not locally symmetric. I also develop the closely related notion of a torsion-free H-structure. The fundamental strategy is to 'classify' the torsion-free connections with a given holonomy H by first 'classifying' the torsion-free H-structures and then examining the problem of determining for any given torsion-free H-structure, its space of compatible torsion-free connections. In nearly all cases, there is a unique compatible torsion-free connection, but there are important exceptions that are closely related to the second-order homogeneous spaces.

I formulate the classification problem for general torsion-free H-structures as a problem treatable by the methods of Cartan-Kähler theory. Finally, I conclude this section with an appendix containing definitions of the various Spencer constructions that will be needed and a discussion of the history of the classification of the irreducible second-order homogeneous spaces. This classification turns out to be important in the classification of the affine torsion-free holonomies in §3.

In §2, I review Berger's list of the possible irreducible holonomies for pseudo-Riemannian metrics which are not locally symmetric. In the course of the review, I analyze each of the possibilities and determine the degree of generality of each one. Among the notable results are, first, that the group $\mathrm{SO}(n,\mathbb{H})$, which appeared on Berger's original list turns out not to be possible as the holonomy of a torsion-free connection, and, second, that there are two extra cases left off the usual lists (see §2.7-8). These can be viewed as alternate real forms of a group whose compact form is $\mathrm{Sp}(p)\cdot\mathrm{Sp}(1)$, the holonomy group of the so-called 'quaternionic-Kähler' metrics.

In §3, I turn to Berger's list of the possible irreducible holonomies for affine connections which are not locally symmetric and do not preserve any non-zero quadratic form. This list turns out to be quite interesting and the examples display a wide variety of phenomena. Actually, one has to remember that Berger's original list was only meant to cover all but a finite number of the possibilities, leaving open the possibility of a finite number of 'exotic' examples. Moreover, in Berger's original list, there was no attempt to deal with the different possibilities for the holonomy of the central part of the group; Berger's classification deals mainly with the classification of the semi-simple part of the irreducible holonomies. It turns out that the center of the group plays a very important role and gives rise to a wealth of examples that had heretofore not been anticipated.

Finally, in §4, I discuss what is known about the exotic examples so far (see Table 4). Perhaps the most interesting of these examples, aside from the examples in dimension 4 first discussed in [Br2], are the ones associated to the 'exceptional' second-order homogeneous spaces of dimension 16 and 27. For example, a consequence of this is that $E_6^{\mathbb{C}} \subset \mathrm{SL}(27,\mathbb{C})$ can occur as the holonomy of a torsion-free (but not locally symmetric) connection on a complex manifold of dimension 27! Unfortunately, as of this writing, the full classification of the possible exotic examples is far from complete.

0.2. Notation. In this report, I have adopted a slightly non-standard nomenclature for the various groups that are to be discussed. This subsection will serve to fix this notation, which is closely related to that used in [KoNa].

I will need to work with vector spaces over $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$. Conjugation has its standard meaning in $\mathbb{C}$ and $\mathbb{H}$; in each case, the fixed subalgebra is $\mathbb{R}$. The symbol $\mathbb{F}$ will be used to denote any one of these division algebras. The elements of the standard n-space $\mathbb{F}^n$ are to be thought of as columns of elements of $\mathbb{F}$ of height n. It is convenient to take all vector spaces over $\mathbb{H}$ to be *right* vector spaces.

For any vector space V over F, the group of invertible $\mathbb{F}$-linear endomorphisms of V will be denoted $\mathrm{GL}(V,\mathbb{F})$ or just $\mathrm{GL}(V)$ when there is no danger of confusion. The algebra of n-by-n matrices with entries in $\mathbb{F}$ will be denoted by $M_n(\mathbb{F})$. This algebra acts on the left of $\mathbb{F}^n$ by the obvious matrix multiplication, representing the algebra $\mathrm{End}_{\mathbb{F}}(\mathbb{F}^n)$. As usual, let $\mathrm{GL}(n,\mathbb{F}) \subset M_n(\mathbb{F})$ denote the Lie group consisting of the invertible matrices in $M_n(\mathbb{F})$, i.e., $\mathrm{GL}(n,\mathbb{F}) = \mathrm{GL}(\mathbb{F}^n)$. When $\mathbb{F} = \mathbb{R}$, the group $\mathrm{GL}(V)$ has two components and it is occasionally useful to use the notation $\mathrm{GL}^+(V)$ for the identity component. For any $A \in M_n(\mathbb{F})$, define $A^* \in M_n(\mathbb{F})$ to be the conjugate transpose of A, so that $(AB)^* = B^* A^*$ for all $A, B \in M_n(\mathbb{F})$.

For a vector space V over $\mathbb{R}$ or $\mathbb{C}$, the notation $\mathrm{SL}(V)$ has its standard meaning. There is no good notion of a quaternionic determinant; however, the obvious identification $\mathbb{H}^n \simeq \mathbb{R}^{4n}$ induces an embedding $\mathrm{GL}(n,\mathbb{H}) \hookrightarrow \mathrm{GL}(4n,\mathbb{R})$ and the subgroup $\mathrm{SL}(n,\mathbb{H}) \subset \mathrm{GL}(n,\mathbb{H})$ is then defined by $\mathrm{SL}(n,\mathbb{H}) = \mathrm{GL}(n,\mathbb{H}) \cap \mathrm{SL}(4n,\mathbb{R})$. Note that $\mathrm{SL}(n,\mathbb{H})$ has codimension 1 (not 4) in $\mathrm{GL}(n,\mathbb{H})$. In Chevalley's nomenclature, $\mathrm{SL}(n,\mathbb{H})$, which is a real form of $\mathrm{SL}(2n,\mathbb{C})$, is denoted $\mathrm{SU}^*(2n)$. My notation for the other real forms of $\mathrm{SL}(n,\mathbb{C})$ are the standard ones: $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SU}(p,q) = \{\, A \in \mathrm{SL}(n,\mathbb{C}) \,|\, A^*\, I_{p,q}\, A = I_{p,q} \,\}$. For simplicity, $\mathrm{SU}(n)$ denotes $\mathrm{SU}(n,0)$.

When $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ and Q is a non-degenerate quadratic form on a vector space V over $\mathbb{F}$, the slightly non-standard usage $\mathrm{SO}(V,Q)$ (respectively, $\mathrm{CO}(V,Q)$) will refer to the identity component of the subgroup of $\mathrm{GL}(V)$ that fixes Q (respectively, that fixes Q up to a multiple). The notations $\mathrm{SO}(p,q)$ $(= \mathrm{SO}(p)$ when $q = 0)$ and $\mathrm{CO}(p,q)$ $(= \mathrm{CO}(p)$ when $q = 0)$ denote the identity components of the standard subgroups of $\mathrm{GL}(p{+}q,\mathbb{R})$, while $\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{CO}(n,\mathbb{C})$ denote the standard subgroups of $\mathrm{GL}(n,\mathbb{C})$. Finally, $\mathrm{SO}(n,\mathbb{H})$ stands for the subgroup consisting of those $A \in \mathrm{GL}(n,\mathbb{H})$ that satisfy $A^*\, iI_n\, A = iI_n$. In Chevalley's nomenclature, $\mathrm{SO}(n,\mathbb{H})$, which is a real form of $\mathrm{SO}(2n,\mathbb{C})$, is denoted $\mathrm{SO}^*(2n)$.

Finally, when $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ and Ω is a non-degenerate skew-symmetric bilinear form on a vector space V over $\mathbb{F}$, the notation $\mathrm{Sp}(V,\Omega)$ (respectively, $\mathrm{CSp}(V,\Omega)$) will stand for the subgroup of $\mathrm{GL}(V)$ that fixes Ω (respectively, that fixes Ω up to a multiple.) The notations $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{CSp}(n,\mathbb{R})$ denote the standard subgroups of $\mathrm{GL}(2n,\mathbb{R})$ while $\mathrm{Sp}(n,\mathbb{C})$ and $\mathrm{CSp}(n,\mathbb{C})$ denote the standard subgroups of $\mathrm{GL}(2n,\mathbb{C})$. (In Chevalley's notation, $\mathrm{Sp}(n,\mathbb{R})$ is denoted by $\mathrm{Sp}^*(n)$.) As for the other real forms of $\mathrm{Sp}(n,\mathbb{C})$, I use the usual $\mathrm{Sp}(p,q)$ to denote the subgroup of $\mathrm{GL}(p{+}q,\mathbb{H})$ consisting of those matrices $A \in M_{p+q}(\mathbb{H})$ that satisfy $A^*\, I_{p,q}\, A = I_{p,q}$, with $\mathrm{Sp}(n,0)$ abbreviated to $\mathrm{Sp}(n)$.

Now define the following subspaces

$$S_n(\mathbb{R}) = \{\ A \in M_n(\mathbb{R}) \bigm| A = {}^tA\ \}$$