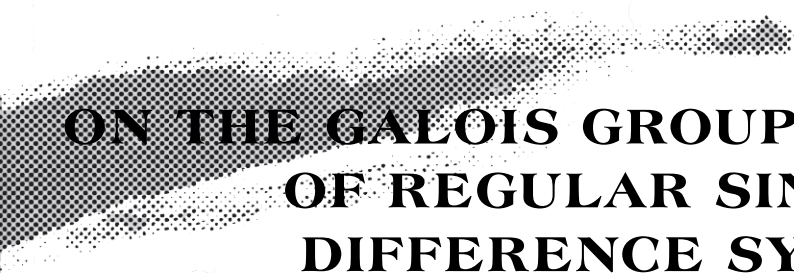


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ON THE GALOIS GROUPS OF FAMILIES OF REGULAR SINGULAR DIFFERENCE SYSTEMS

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Abstract. — We investigate the variation of the dimension of the Galois groups of families of regular singular difference systems using analytic tools.

Résumé (Sur les groupes de Galois des familles de systèmes aux différences réguliers singuliers)

Nous étudions la variation de la dimension des groupes de Galois de familles de systèmes aux différences singuliers réguliers au moyen d'outils analytiques.

1. Introduction-Organization

1.1. Introduction-Main results. — In the whole paper, x will denote a *complex* variable and τ (resp. δ) will denote the difference operator acting on a function Y of the complex variable x by $\tau Y(x) = Y(x-1)$ (resp. $\delta Y(x) = (x-1)(Y(x) - Y(x-1))$).

Let us consider :

$$(\mathcal{S}_h) : \tau Y = A_h Y, \quad A_h \in \mathrm{GL}_n(\mathbb{C}(x, h))$$

a family of regular singular difference systems parameterized by $h \in \mathbb{C} \setminus \Sigma$, Σ being a finite subset of $\mathbb{C}$, and let us denote by G_h the corresponding difference Galois groups over $\mathbb{C}(x)$ (see [11]). Note that, in what follows, the algebraic dependence on h is not essential and could have been replaced by analytic dependence on h on some open subset of $\mathbb{C}$.

In this paper we study the variation of $\dim G_h$ (dimension of the complex linear algebraic group G_h) with respect to h via an analytic approach.

Let us recall that $(\mathcal{S}_h)$ is Fuchsian if $A_h(\infty) = I_n$, in which case we set $A_{h;\infty} = \lim_{x \rightarrow \infty} (x-1)(I_n - A_h(x)) \in M_n(\mathbb{C})$. It is moreover nonresonant if,

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for any pair (λ, μ) of distinct eigenvalues of $A_{h;\infty}$, we have $\lambda - \mu \notin \mathbb{Z}$. The system (S_h) is regular singular if there exists $F_h \in \mathrm{GL}_n(\mathbb{C}(x))$ such that the system defined by $F_h[A_h] := (F_h(x-1))^{-1}A_h(x)F_h(x)$ is Fuchsian. For details about these classical notions, we refer for instance to sections 1 to 4 of [14] and to chapter 9 of [11] and to the references therein.

The regular singular difference systems are classified by their Birkhoff connection matrices; this is in some sense similar to the classification of the regular singular differential systems by means of their monodromy representations. For any $h \in \mathbb{C} \setminus \Sigma$, we associate to (S_h) its Birkhoff connection matrix $P_h \in \mathrm{GL}_n(\mathbb{C}(\mathbf{x}))$ with $\mathbf{x} = e^{2\pi i x}$ (see section 2). These give rise, for any $h \in \mathbb{C} \setminus \Sigma$, to a family of Galoisian morphisms $\Lambda_h(a, b) := (P_h(a))^{-1}P_h(b) \in G_h$ parameterized by all $(a, b) \in \mathbb{C}^2$ such that $P_h(a)$ and $P_h(b)$ are defined and invertible (this was pointed out for the first time by P. Etingof in [7] in the case of regular q -difference systems and extended to regular singular (q -)difference systems by M. Van der Put and M. Singer in [11]; for a different, more “analytic”, point of view in the q -difference case we refer to the work of J. Sauloy in [16]). These morphisms allow us to give a group-theoretic description of the Galois groups G_h -see Theorem 3.2 in section 3.1- (the fact that they generate Zariski-dense subgroups of the Galois groups is proved in [11]) and of their Lie algebras $\mathfrak{g}_h$ -see Theorem 3.3 in section 3.2-.

Using the above-mentioned description of $\mathfrak{g}_h$, we prove, in section 5, under the hypotheses 1. to 3. stated in section 4.1, that :

Theorem. — *Let $\kappa = \max_{h \in \mathbb{C} \setminus \Sigma} \dim(G_h)$. Then $\Theta = \{h \in \mathbb{C} \setminus \Sigma \mid \dim(G_h) = \kappa\}$ is an open subset of $\mathbb{C} \setminus \Sigma$ with discrete complement.*

The following result is an immediate consequence of the above Theorem.

Corollary. — *Suppose that there exists $h \in \mathbb{C} \setminus \Sigma$ such that $G_h = \mathrm{GL}_n(\mathbb{C})$. Then for any $h \in \mathbb{C} \setminus \Sigma$ but, maybe, a discrete subset, we have $G_h = \mathrm{GL}_n(\mathbb{C})$.*

Note that, replacing x by $\frac{x}{h}$, we can make the parameter h be also the step of the equation. We leave the corresponding statements to the reader.

In a different context, the idea of an analytic approach for the study of the variation of Galois groups appears in the work of J. Sauloy in [15, 16] and is an essential motivation for A. Duval and the author’s papers [4, 5, 6, 14]; see also L. Di Vizio and C. Zhang’s paper [2]. The reader will find more informations about the algebraic meaning of the analytic theory of (q -)difference equations in the works of P. Etingof [7], of J.-P. Ramis and J. Sauloy [12, 13], of J. Sauloy [16] and of M. Singer and M. Van der Put [11]. Moreover, for problems and results related to the main subject of the present paper, we refer the reader to Y. André’s paper [1]. Concerning parameterized q -difference equations, we also refer to section 5 of C. Hardouin and M. Singer’s paper [8].

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1.2. Organization. — In section 2 we recall useful properties of the regular singular difference systems. In section 3 we give a group-theoretic description of the Galois group of a given regular singular difference system and of its Lie algebra in terms of a corresponding Birkhoff matrix. In section 4 we consider a family of regular singular difference systems parameterized by $h \in \mathbb{C} \setminus \Sigma$ and we study the dependence of corresponding Birkhoff matrices on the parameter h . In section 5 we prove our main theorem concerning the variation of the dimension of the Galois groups.

2. Regular singular systems : a reminder

2.1. Factorial series. — For the material presented in this section, we refer to section 2.1. of [3] and to [10].

A function a defined and holomorphic on some open subset of $\mathbb{C}$ containing some half-plane $\Pi_M^+ := \{x \in \mathbb{C} \mid \Re(x) > M\}$ is expandable into a factorial series on Π_M^+ if a admits an expansion, convergent on Π_M^+ , of the form :

$$\sum_{s=0}^{+\infty} a_s x^{-[s]}$$

where, for all $s \in \mathbb{N}$,

$$x^{-[s]} = \frac{1}{x(x+1) \cdots (x+s-1)}.$$

When it exists, the factorial series expansion is unique. For later use, for all $s \in \mathbb{N}$, we also introduce the following notation :

$$x^{[s]} = x(x+1) \cdots (x+s-1).$$

The set of germs of holomorphic functions expandable into factorial series, denoted by $\mathcal{O}_{fact}$, is by definition the direct limit of the sets of holomorphic functions expandable into factorial series on the half plane Π_M^+ as M tends to $+\infty$ (in what follows, we will identify an element of the direct limit with one of its representatives). It is a subring of the ring of germs of holomorphic functions at $+\infty$ which is, by definition, the direct limit of the rings of functions holomorphic on the half plane Π_M^+ as M tends to $+\infty$. In particular $\mathcal{O}_{fact}$ is an integral domain; its field of fractions is denoted by $\mathcal{M}_{fact}$. The intersection of $\mathcal{O}_{fact}$ and $\mathcal{M}_{fact}$ with $\mathcal{M}(\mathbb{C})$, the field of meromorphic functions over $\mathbb{C}$, are respectively denoted by $\mathcal{O}_{fact}(\mathbb{C})$ and $\mathcal{M}_{fact}(\mathbb{C})$.

Replacing x by $-x$, we get the notion of function expandable into retrofactorial series. More explicitly, a function a holomorphic on some open subset of $\mathbb{C}$ containing

some half-plane $\Pi_M^- := \{x \in \mathbb{C} \mid \Re(x) < M\}$ is expandable into a retrofactorial series on Π_M^- if a admits an expansion, convergent on Π_M^- , of the form :

$$\sum_{s=0}^{+\infty} a_s x^{-[s]}$$

where, for all $s \in \mathbb{N}$,

$$x^{-[s]} = \frac{1}{x(x-1)\cdots(x-s+1)}.$$

When it exists, the retrofactorial series expansion is unique. For later use, for all $s \in \mathbb{N}$, we introduce the notation :

$$x^{[s]} = x(x-1)\cdots(x-s+1).$$

Moreover, we introduce the rings and fields of retrofactorial series $\mathcal{O}_{\text{retrofact}}$, $\mathcal{M}_{\text{retrofact}}$, $\mathcal{O}_{\text{retrofact}}(\mathbb{C})$ and $\mathcal{M}_{\text{retrofact}}(\mathbb{C})$ defined similarly to $\mathcal{O}_{\text{fact}}$, $\mathcal{M}_{\text{fact}}$, $\mathcal{O}_{\text{fact}}(\mathbb{C})$ and $\mathcal{M}_{\text{fact}}(\mathbb{C})$ respectively.

For instance, any function defined and analytic in a neighborhood of $\infty \in \mathbb{P}_{\mathbb{C}}^1$ is expandable into factorial series and retrofactorial series.

We will denote by $\widehat{\mathcal{O}}_{\text{fact}}$ the integral domain of formal factorial series and we denote by $\widehat{\mathcal{M}}_{\text{fact}}$ its field of fractions. The ring laws on $\widehat{\mathcal{O}}_{\text{fact}}$ are given, for all $a(x) = \sum_{s=0}^{+\infty} a_s x^{-[s]} \in \widehat{\mathcal{O}}_{\text{fact}}$ and $b(x) = \sum_{s=0}^{+\infty} b_s x^{-[s]} \in \widehat{\mathcal{O}}_{\text{fact}}$, by :

$$(1) \quad (a+b)(x) = \sum_{s=0}^{+\infty} (a_s + b_s) x^{-[s]}$$

and :

$$(2) \quad (ab)(x) = \sum_{s=0}^{+\infty} c_s x^{-[s]}$$

where :

$$c_0 = a_0 b_0 \text{ and, } \forall s \in \mathbb{N}^*, \quad c_s = a_0 b_s + a_s b_0 + \sum_{(j,k,l) \in J_s} c_{j,l}^{(k)} a_j b_l$$

with :

$$(3) \quad \forall s \in \mathbb{N}^*, \quad J_s = \{(j, k, l) \mid j, l \geq 1, k \geq 0, j+k+l=s\}$$

and :

$$(4) \quad \forall (j, l) \in \mathbb{N}^* \times \mathbb{N}^*, \forall k \in \mathbb{N}, \quad c_{j,l}^{(k)} = \frac{(j+k-1)!(l+k-1)!}{k!(j-1)!(l-1)!}.$$

As above, replacing x by $-x$, we get the integral domain of formal retrofactorial series $\widehat{\mathcal{O}}_{\text{retrofact}}$; its field of fractions is denoted by $\widehat{\mathcal{M}}_{\text{retrofact}}$.

We can interpret any element A of $M_{n,m}(\mathcal{O}_{\text{fact}})$ or $M_{n,m}(\widehat{\mathcal{O}}_{\text{fact}})$ as a series $\sum_{s=0}^{+\infty} A_s x^{-[s]}$ with coefficients in $M_{n,m}(\mathbb{C})$. The above sum and product formulas