

Séminaires & Congrès

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ELLIPTIC HYPERGEOMETRIC TERMS

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**ARITHMETIC AND GALOIS THEORY
OF DIFFERENTIAL EQUATIONS**

Numéro 23

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Abstract. — General structure of the multivariate plain and q -hypergeometric terms and univariate elliptic hypergeometric terms is described. Some explicit examples of the totally elliptic hypergeometric terms leading to multidimensional integrals on root systems, either computable or obeying non-trivial symmetry transformations, are presented.

Résumé (Termes elliptiques hypergéométriques). — Nous décrivons la structure générale des termes hypergéométriques usuels et q -hypergéométriques de plusieurs variables, ainsi que le terme hypergéométrique elliptique d'une variable. Nous présentons des exemples de termes hypergéométriques totalement elliptiques conduisant à des intégrales multidimensionnelles sur des systèmes de racines, qui soit sont calculables soit obéissent à des symétries non triviales.

1. Plain hypergeometric case

The definition of the hypergeometric series goes as far back as to Euler and, in a more general setting, to Pochhammer and Horn [1, 7].

Definition 1. — The formal series

$$\sum_{m \in \mathbb{Z}^n} c(m) = \sum_{m \in \mathbb{Z}^n} c(m_1, \dots, m_n)$$

is called plain hypergeometric series, if the ratios

$$\frac{c(m_1, \dots, m_i + 1, \dots, m_n)}{c(m_1, \dots, m_n)} = R_i(m_1, \dots, m_n)$$

2000 Mathematics Subject Classification. — 33E20, 33E05.

Key words and phrases. — Elliptic hypergeometric functions, elliptic gamma functions, elliptic beta integrals.

Lectures presented at the Workshop "Théories galoisiennes et arithmétiques des équations différentielles" (September 2009, CIRM, Luminy, France).

are rational functions of m_1, \dots, m_n .

Suppose that given rational functions $R_i(m)$, called certificates, satisfy the consistency conditions

$$R_i(m_1, \dots, m_k + 1, \dots, m_n)R_k(m) = R_k(m_1, \dots, m_i + 1, \dots, m_n)R_i(m).$$

The general form of corresponding (admissible) plain hypergeometric series was determined by Ore and Sato (e.g., see the survey [7]).

Theorem 1. — *General admissible plain hypergeometric terms $c(m)$ have the form*

$$c(m) = \frac{R(m)}{\prod_{j=1}^K \Gamma(\epsilon_j(m) + a_j)} \prod_{k=1}^n z_k^{m_k},$$

where z_k, a_j are arbitrary complex parameters, $K = 0, 1, 2, \dots$, $\epsilon_j(m) = \sum_{k=1}^n \epsilon_{jk} m_k$, $\epsilon_{jk} \in \mathbb{Z}$, $R(m)$ is some rational function of m_1, \dots, m_n , and $\Gamma(x)$ is the standard Euler gamma function.

Using the inversion formula $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$, some of the Γ -functions can be put from the denominator of $c(m)$ to its numerator.

In a similar way one can treat hypergeometric integrals [18].

Definition 2. — *The integrals*

$$\int_D \Delta(x) dx = \int_D \Delta(x_1, \dots, x_n) dx_1 \dots dx_n,$$

for some domain of integration $D \in \mathbb{C}^n$, are called plain hypergeometric integrals, if the ratios

$$\frac{\Delta(x_1, \dots, x_i + 1, \dots, x_n)}{\Delta(x_1, \dots, x_n)} = R_i(x_1, \dots, x_n)$$

are rational functions of x_1, \dots, x_n .

The general admissible plain hypergeometric terms $\Delta(x)$ have the form

$$\Delta(x) = \varphi(x)R(x) \frac{\prod_{j=1}^K \Gamma(\sum_{k=1}^n \mu_{jk} x_k + b_j)}{\prod_{j=1}^M \Gamma(\sum_{k=1}^n \epsilon_{jk} x_k + a_j)} \prod_{k=1}^n z_k^{x_k},$$

where z_k, a_j, b_j are arbitrary complex parameters, $K, M = 0, 1, 2, \dots$, $\mu_{jk}, \epsilon_{jk} \in \mathbb{Z}$, $R(x)$ is some rational function of x_1, \dots, x_n , and $\varphi(x) = \varphi(x_1, \dots, x_i + 1, \dots, x_n)$ is an arbitrary periodic function. The plain hypergeometric series can be obtained from integrals as sums of residues for particular sequences of poles of $\Delta(x)$.

The $\Gamma(x)$ -function can be defined as a special meromorphic solution of the functional equation

$$\Gamma(x + 1) = x\Gamma(x),$$

the general solution of which has the form $\varphi(x)\Gamma(x)$, where $\varphi(x + 1) = \varphi(x)$ is an arbitrary periodic function.

For $n = 1$, definition 2 yields the Meijer function for the choices $\varphi = 1$, $R = 1$ and D one of the contours 1) $\{-i\infty, +i\infty\}$ separating sequences of equidistant poles going

to the left and to the right of the complex plane, 2) $\{-\infty - iA, -\infty + iB\}$ encircling sequences of poles going to the left (for some choice of the positive constants A and B), 3) $\{+\infty - iA, +\infty + iB\}$ encircling sequences of poles going to the right (for some choice of the constants A and B).

It is worth to remark that a limiting form of the plain hypergeometric terms is determined by the system of partial differential equations

$$\frac{1}{\Delta(x)} \frac{\partial \Delta(x)}{\partial x_i} = R_i(x).$$

2. q -hypergeometric case

q -deformations of hypergeometric functions were introduced by Heine a long time ago [1].

Definition 3. — *The formal series*

$$\sum_{m \in \mathbb{Z}^n} c(m) = \sum_{m \in \mathbb{Z}^n} c(m_1, \dots, m_n)$$

is called q -hypergeometric, if

$$\frac{c(m_1, \dots, m_i + 1, \dots, m_n)}{c(m_1, \dots, m_n)} = R_i(q^{m_1}, \dots, q^{m_n})$$

are rational functions of q^{m_1}, \dots, q^{m_r} , where q is an arbitrary complex parameter.

This is a natural extension of the previous definition and it leads [7] to the following theorem.

Theorem 2. — *General admissible q -hypergeometric terms $c(m)$ have the form*

$$c(m) = R(q^m) \frac{\prod_{j=1}^K (a_j; q)_{\mu_j(m)}}{\prod_{j=1}^M (b_j; q)_{\epsilon_j(m)}} \prod_{k=1}^n x_k^{m_k},$$

where x_k, a_j, b_j are arbitrary complex parameters, $K, M = 0, 1, 2, \dots$, $\mu_j(m) = \sum_{k=1}^n \mu_{jk} m_k$ and $\epsilon_j(m) = \sum_{k=1}^n \epsilon_{jk} m_k$ with $\mu_{jk}, \epsilon_{jk} \in \mathbb{Z}$, $R(q^m)$ is some rational function, and

$$(x; q)_n := \begin{cases} \prod_{j=0}^{n-1} (1 - xq^j), & \text{for } n > 0 \\ \prod_{j=1}^{-n} (1 - xq^{-j})^{-1}, & \text{for } n < 0 \end{cases}$$

is the q -shifted factorial (or the q -Pochhammer symbol).

Definition 4. — *The integrals*

$$\int_D \Delta(x) dx = \int_D \Delta(x_1, \dots, x_n) dx_1 \dots dx_n,$$

for some domain of integration $D \in \mathbb{C}^n$, are called q -hypergeometric, if the ratios

$$\frac{\Delta(x_1, \dots, x_i + 1, \dots, x_n)}{\Delta(x_1, \dots, x_n)} = R_i(q^{x_1}, \dots, q^{x_n})$$

are rational functions of q^{x_1}, \dots, q^{x_n} .

Define q -gamma functions as special meromorphic solutions of the finite-difference equation

$$(2.1) \quad f(u + \omega_1) = (1 - e^{2\pi i u / \omega_2})f(u),$$

where $\omega_1, \omega_2 \in \mathbb{C}$. Evidently, solutions of this equation are defined modulo multiplication by an arbitrary $\varphi(u + \omega_1) = \varphi(u)$ periodic function. Introducing the variables

$$q := e^{2\pi i \omega_1 / \omega_2}, \quad z := e^{2\pi i u / \omega_2},$$

this equation can be replaced by

$$\Gamma_q(qz) = (1 - z)\Gamma_q(z).$$

For $|q| < 1$ its particular solution, analytic near the point $z = 0$, is determined by a simple iteration, which yields

$$\Gamma_q(z) = \frac{1}{(z; q)_\infty} = \prod_{j=0}^\infty \frac{1}{1 - zq^j}, \quad \Gamma_q(0) = 1.$$

This function can be considered as a q -gamma function for $|q| < 1$. More precisely, the Thomae-Jackson q -gamma function has the form

$$\Gamma(u; q) = (1 - q)^{1-u} \frac{(q; q)_\infty}{(q^u; q)_\infty}, \quad \lim_{q \rightarrow 1} \Gamma(u; q) = \Gamma(u).$$

It satisfies the equations

$$\Gamma(u + 1; q) = \frac{1 - q^u}{1 - q} \Gamma(u; q), \quad \Gamma(u - \frac{2\pi i}{\log q}; q) = (1 - q)^{\frac{2\pi i}{\log q}} \Gamma(u; q).$$

For $|q| < 1$ the q -Pochhammer symbol can be written as

$$(t; q)_n = \frac{(t; q)_\infty}{(tq^n; q)_\infty}, \quad n \in \mathbb{Z}.$$

For $|q| = 1$ the equation $f(qz) = (1 - z)f(z)$ does not have meromorphic solutions for $z \in \mathbb{C}^*$. In this case it is necessary to consider equation (2.1) and search for its solutions meromorphic in $u \in \mathbb{C}$. The modified q -gamma function

$$(2.2) \quad \gamma(u; \omega_1, \omega_2) = \exp\left(-\int_{\mathbb{R}+i0} \frac{e^{ux}}{(1 - e^{\omega_1 x})(1 - e^{\omega_2 x})} \frac{dx}{x}\right),$$

where the contour $\mathbb{R} + i0$ passes along the real axis turning over the point $x = 0$ from above in an infinitesimal way, solves (2.1) and remains meromorphic for $\omega_1, \omega_2 > 0$ when $|q| = 1$. (2.2) is known as the Barnes-Shintani “double sine” function, the noncompact quantum dilogarithm, or the hyperbolic gamma function (see the survey [14] for relevant references).

We assume that $\text{Re}(\omega_1), \text{Re}(\omega_2) > 0$. Then the integral (2.2) is convergent for $0 < \text{Re}(u) < \text{Re}(\omega_1 + \omega_2)$. Under appropriate restrictions on u and $\omega_{1,2}$, it can be