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INTERSECTION THEORY ON PUNCTUAL HILBERT SCHEMES AND GRADED HILBERT SCHEMES

by

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Abstract. — The rational Chow ring $A^*(S^{[n]}, \mathbb{Q})$ of the Hilbert scheme $S^{[n]}$ parametrising the length n zero-dimensional subschemes of a toric surface S can be described with the help of equivariant techniques. In this paper, we explain the general method and we illustrate it through many examples. In the last section, we present results on the intersection theory of graded Hilbert schemes.

***Résumé* (Théorie de l'intersection sur les schémas de Hilbert ponctuels et gradués)**
Les techniques équivariantes permettent de décrire l'anneau de Chow rationnel $A^*(S^{[n]}, \mathbb{Q})$ du schéma de Hilbert $S^{[n]}$ paramétrant les sous-schémas ponctuels de longueur n d'une surface torique S. Dans cet article, nous présentons la démarche générale et nous l'illustrons au travers de nombreux exemples. La dernière section expose des résultats de théorie d'intersection sur des schémas de Hilbert gradués.

Introduction

Let S be a smooth projective surface and $S^{[n]}$ the Hilbert scheme parametrising the length n zero-dimensional subschemes of S. How to describe the cohomology ring $H^*(S^{[n]}, \mathbb{Q})$ and the Chow ring $A^*(S^{[n]}, \mathbb{Q})$?

A first approach is based on the work of Nakajima, Grojnowski and Lehn among others [12], [16], [13], [14], [2]. The direct sum $\oplus_{n \in \mathbb{N}} H^*(S^{[n]}, \mathbb{Q})$ is an (infinite dimensional) irreducible representation and carries a Fock space structure [15]. Lehn settles a connection between the Fock space structure and the intersection theory of the Hilbert scheme via the action of the Chern classes of tautological bundles [11].

An other method, independent of the Fock space formalism introduced by Nakajima, has been developed in [7] when S is a toric surface. The point is that the extra structure coming from the torus action brings into the scene an equivariant Chow ring

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which is easier to compute than the classical Chow ring. The classical Chow ring is a quotient of the equivariant Chow ring.

The computations of this equivariant approach are explicit. They rely on the standard description of the cohomology of the Grassmannians and on a description of the tangent space to the Hilbert scheme at fixed points.

The main goal of this paper is to present this equivariant approach. We follow the general theory and we illustrate it with the case $S = \mathbb{P}^2$ and $n = 3$ as the main example.

In the last section, we bring our attention to graded Hilbert schemes, which played an important role in the equivariant computations. We present results on the set theoretic intersection of Schubert cells, which suggest that intersection theory on graded Hilbert schemes could be described in terms of combinatorics of plane partitions.

Throughout the paper, we use the formalism of Chow rings and work over any algebraically closed field k. When $k = \mathbb{C}$, the Chow ring coïncides with usual cohomology since the action of the two-dimensional torus T on S induces an action of T on $S^{[d]}$ with a finite number of fixed points.

1. Equivariant intersection theory

1.1. General results. — In this section, we recall the facts about equivariant Chow rings that we need. To simplify the presentation, we work with rational coefficients and the notation $A^*(X) := A^*(X, \mathbb{Q})$ denotes the rational Chow ring.

The construction of an equivariant Chow ring associated with an algebraic space endowed with an action of a linear algebraic group has been settled by Edidin and Graham [3]. Their construction is modeled after the Borel construction in equivariant cohomology.

Proposition 1. — *Let G be an algebraic group, X an equidimensional quasi-projective scheme with a linearized G-action and $i, j \in \mathbb{Z}$, $i \leq dim(X), j \geq 0$. There exists a representation V of G such that*

- *V contains an open set U on which G acts freely,*
- *$U \to U/G$ exists as a scheme and is a principal G bundle,*
- $\operatorname{codim}_V V \setminus U > \dim(X) - i$.

The quotient $X_G = (X \times U)/G$ under the diagonal action exists as a scheme. The groups $A_i^G(X) := A_{i+\dim(V)-\dim(G)}(X_G)$ and $A_G^j(X) := A^G_{\dim(X)-j}(X)$ are independent of the choice of the couple (U, V).

Definition 2. — *The group $A_G^i(X)$ is by definition the equivariant Chow group of X of degree i.*

***Example* 3.** — *If $G = T = (k^*)^n$ is a torus, then a possible choice for the couple (U, V) is $V = (k^l)^n$ with $l \gg 0$, and $U = (k^l - \{0\})^n$ with T acting on V by $(t_1, \ldots, t_n)(x_1, \ldots, x_n) = (t_1x_1, \ldots, t_nx_n)$. The quotient U/T is isomorphic to $(\mathbb{P}^{l-1})^n$.*

***Example* 4.** — *Let p be a point and $T = (k^*)^n$ the torus acting trivially on p. Then $A_T^*(p) \simeq \mathbb{Q}[h_1, \ldots, h_n]$ where h_i has degree 1 for all i.*

Proof. — By the above example, $A_T^*(p) = \lim_{l\to\infty} A^*((\mathbb{P}^{l-1})^n) = \lim_{l\to\infty} \mathbb{Q}[h_1, \ldots, h_n]/(h_1^l, \ldots, h_n^l) = \mathbb{Q}[h_1, \ldots, h_n]$, where h_i has degree 1 (according to the definition of the equivariant Chow group, the limit considered is a degreewise stabilisation thus the limit is the polynomial ring and not a power series ring). ☐

If X is smooth, then $(X \times U)/G$ is smooth too and $A_T^*(X)$ is a ring: the intersection of two classes $u, v \in A_T^*(X)$ takes place in the Chow ring $A^*((X \times U)/G)$.

***Example* 5.** — *The isomorphism $A_T^*(p) \simeq \mathbb{Q}[h_1, \ldots, h_n]$ of the last example is an isomorphism of rings.*

***Definition* 6.** — *Let E be a G-equivariant vector bundle on X and $E_G \to X_G$ the vector bundle with total space $E_G = (E \times U)/G$. The equivariant Chern class $c_j^G(E)$ is defined by $c_j^G(E) = c_j(E_G) \in A^j(X_G) = A_G^j(X)$.*

The identification of $A_T^*(p)$ with a ring of polynomials R can be made intrinsic using equivariant Chern classes.

***Proposition* 7.** — *Let $\hat{T}$ be the character group of a torus $T \simeq (k^*)^n$. Any character $\chi \in \hat{T}$ defines a one-dimensional representation of T by $t.k = \chi(t)k$, hence an equivariant bundle over the point and an equivariant Chern class $c_1^T(\chi)$. The map $\chi \to c_1^T(\chi) \in A_T^1(p)$ extends to an isomorphism $R = Sym_{\mathbb{Q}}(\hat{T}) \to A_T^*(p)$, where $Sym_{\mathbb{Q}}(\hat{T})$ is the symmetric algebra over $\mathbb{Q}$ of the group $\hat{T}$.*

***Example* 8.** — *Let $T = k^*$ be the one dimensional torus acting on the projective space $\mathbb{P}^r = Proj\ k[x_0, \ldots, x_r]$ by $t.(x_0 : \ldots : x_r) = (t^{n_0}x_0 : \ldots : t^{n_r}x_r)$. Then $A_T^*(\mathbb{P}^r) = \mathbb{Q}[t, h]/p(h, t)$ where $p(h, t) = \sum_{i=0}^r h^{r-i} e_i(n_0t, \ldots, n_rt)$, e_i being the i-th elementary symmetric polynomial.*

Proof. — X_T is the $\mathbb{P}^r$ bundle $\mathbb{P}(\mathcal{O}(n_0) \oplus \cdots \oplus \mathcal{O}(n_r))$ over $\mathbb{P}^{l-1}$. The rational Chow ring of this projective bundle is $\mathbb{Q}[h, t]/(p(h, t), t^l)$. We have the result when l tends to ∞. ☐

***Example* 9.** — *Let V be a representation of G and $G(k, V)$ the corresponding Grassmannian. Then $A_G^*(G(k, V))$ is generated as an R module by the equivariant Chern classes of the universal quotient bundle.*

Proof. — The quotient $(G(k,V)\times U)/G$ is a Grassmann bundle over U/G with fiber isomorphic to $G(k,V)$. Since the Chow rings of Grassmann bundles are generated over the Chow ring of the base by the Chern classes of the universal quotient bundle, the result follows. □

1.2. Results specific to the action of tori. — Brion [1] pushed further the theory of equivariant Chow rings when the group is a torus T acting on a variety X.

Theorem 10. — [1] *Let X be a smooth projective T-variety. The restriction morphism $i_T^*: A_T^*X \to A_T^*X^T$ is injective.*

Example 11. — *Let $T=k^*$ act on $\mathbb{P}^1$ by $t.(x:y)=(tx:y)$. The inclusion $i_T^*: A_T^*(\mathbb{P}^1)\to A_T^*(\{0,\infty\})=R^2$ identifies $A_T^*(\mathbb{P}^1)$ with the couples (P,Q) of polynomials $\in R=\mathbb{Q}[t]$ such that $P(0)=Q(0)$.*

Proof. — Let $V=Vect(x,y)$ be the 2 dimensional vector space with $\mathbb{P}(V)=\mathbb{P}^1$. By the above $A_T^*(\mathbb{P}^1)$ is generated by the Chern classes of the universal quotient bundle as an R-module. On the point $\infty = ky\in\mathbb{P}(V)$, the quotient bundle Q is isomorphic to kx and T acts with character t. Thus $c_1(Q)_\infty = t$. Similarly, the restriction of Q to the point $0=kx$ is a trivial equivariant bundle and $c_1(Q)_0=0$. Thus $c_1(Q)$ restricted to $\{0,\infty\}$ is $(0,t)$. Obviously, $c_0(Q)=(1,1)$. Thus $A_T^*(\mathbb{P}^1)=\mathbb{Q}[t](0,t)+\mathbb{Q}[t](1,1)$ as expected. □

If $T'\subset T$ is a one codimensional torus, the localisation morphism i_T^* factorizes: $A_T^*(X)\to A_T^*(X^{T'})\overset{i_{T'}^*}{\to} A_T^*(X^T)=R^{X^T}$. Brion has shown

Theorem 12. — [1] *Let X be a smooth projective variety with an action of T. The image $Im(i_T^*)$ satisfies $Im(i_T^*)=\cap_{T'}Im(i_{T'}^*)$ where the intersection runs over all subtori T' of codimension one in T.*

An important point is that the equivariant Chow groups determine the usual Chow groups. The fibers of $X_T\to U/T$ are isomorphic to X. Let $j: X\to X_T$ be the inclusion of a fiber and $j^*: A_T^*(X)\to A^*(X)$ the corresponding restriction.

Theorem 13. — [1] *Let $R^+=\hat{T}R\subset R$ be the set of polynomials with positive valuation. The morphism j^* is surjective with kernel $R^+A_T^*(X)$.*

Example 14. — $A^*(\mathbb{P}^1) = (\mathbb{Q}[t](t,0)+\mathbb{Q}[t](1,1))/(\mathbb{Q}[t]^+(t,0)+\mathbb{Q}[t]^+(1,1)) \simeq \mathbb{Q}[t]/(t^2)$. *The isomorphism sends $(P=\sum p_it^i, Q=\sum q_it_i)$ with $p_0=q_0$ to (p_0,p_1-q_1).*

Finally, we have an equivariant Kunneth formula for the restriction to fixed points, proved in [7].