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ON THE CODIFFERENCE OF LINEAR FRACTIONAL STABLE MOTION

by

Joshua B. Levy & Murad S. Taqqu

Abstract. — This paper extends a previous study on the dependence configuration for linear fractional stable noise (LFSN), an α-stable, H-self-similar stationary increment (H-sssi) process defined for $0 < \alpha < 2$ and $0 < H < 1, H \neq 1/\alpha$. A measure of dependence, valid for any stationary process Y, is $r_Y(\theta_1, \theta_2; t) = \mathbb{E}e^{i(\theta_1 Y(t)+\theta_2 Y(0))} - \mathbb{E}e^{i(\theta_1 Y(t))}\mathbb{E}e^{i(\theta_2 Y(0))}$ with $\theta_1, \theta_2 \in \mathbb{R}$ and $t \in \mathbb{R}$. Since LFSN is a moving average, $r_Y(t) \to 0$ as $t \to \infty$ for all (θ_1, θ_2). How fast is the convergence? The speed with which $r_Y(t)$ tends to zero has been studied in several papers. The rates depend on the specific values of α and H. This paper treats a previously unexamined boundary case where $1 < \alpha < 2$ and $H = 1 - 1/(\alpha(\alpha - 1))$.

***Résumé* (Sur la co-différence des mouvements fractionnaires linéaires, stables).** — Cet article étend une étude antérieure sur la co-différence du bruit fractionnaire linéaire, stable (LFSN) qui est un processus α-stable, H-auto-similaire et à accroissments stationnaires (H-sssi), avec $0 < \alpha < 2$ at $0 < H < 1, H \neq 1/\alpha$. Une mesure de dépendance valide pour tout processus stationnaire Y est donnée par $r_Y(\theta_1, \theta_2; t) = \mathbb{E}e^{i(\theta_1 Y(t)+\theta_2 Y(0))} - \mathbb{E}e^{i(\theta_1 Y(t))}\mathbb{E}e^{i(\theta_2 Y(0))}$, avec $\theta_1, \theta_2 \in \mathbb{R}$ et $t \in \mathbb{R}$. Puisque le LFSN est une moyenne mobile, $r_Y(t) \to 0$ as $t \to \infty$ pour tous (θ_1, θ_2). Quelle est alors la vitesse de convergence ? Celle-ci, lorsque $r_Y(t)$ tend vers 0, a été étudiée dans plusieurs articles. Les différents régimes, dépendent des valeurs de α et H. Cet article traite d'un cas non encore étudié ; celui pour lequel $1 < \alpha < 2$ et $H = 1 - 1/(\alpha(\alpha - 1))$.

1. Introduction

This research is a further study on the dependence structure of *linear fractional stable motion* (LFSM), which is a self-similar process with self-similarity exponent H. The main result is Theorem 3.1, which provides the asymptotic behavior of a measure of dependence, known as the codifference, of the increments of LFSM in a

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critical case $H = \overline{H}$. We will recall in Section 2 what happens when H is not equal to the critical value $\overline{H}$.

First, however, is some background. Denoted here by $X_{\alpha,H}(a,b) = \{X_{\alpha,H}(a,b;t)\}, t \in \mathbb{R}$, LFSM is defined by the stochastic integral

$$X_{\alpha,H}(a,b;t) := \int_{-\infty}^{\infty} f_{\alpha,H}(a,b;t,x) M(\mathrm{d}x). \tag{1}$$

The parameters α and H satisfy

$$0 < \alpha < 2 \quad \text{and} \quad 0 < H < 1, H \neq 1/\alpha,$$

while the "linear" (or "spatial") coefficients a and b are real numbers such that $|a|+|b| > 0$. M is an α-stable random measure with control space $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$, and with control measure Lebesgue measure $m(\mathrm{d}x) = \mathrm{d}x$ and *constant* skewness intensity, $\beta : \mathbb{R} \to [-1, 1]$, such that $\beta(x) = \beta$ and $\beta = 0$ when $\alpha = 1$. The kernel $f_{\alpha,H}$ is defined by

$$f_{\alpha,H}(a,b;t,x) := a\left[(t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}}\right] + b\left[(t-x)_-^{H-\frac{1}{\alpha}} - (-x)_-^{H-\frac{1}{\alpha}}\right]. \tag{2}$$

For $y, p \in \mathbb{R}$ $y_+ = \max\{y, 0\}$, $y_- = -\min\{y, 0\}$ and $y_+^{\mathrm{p}} = (y_+)^{\mathrm{p}}$, $y_-^{\mathrm{p}} = (y_-)^{\mathrm{p}}$ with the modification $y_+^{\mathrm{p}} = 0$ if $y \leq 0$, $y_-^{\mathrm{p}} = 0$ if $y \geq 0$. Since (as is checked easily) $\int_{-\infty}^{\infty} |f_{\alpha,H}(a,b;t,x)|^\alpha \mathrm{d}x < \infty$, LFSM is well-defined.

LFSM is an α-stable process. In particular, it is "heavy-tailed" because its marginal distributions decay hypergeometrically like $x^{-\alpha}$ as $x \to \infty$. This impels moments of order less than α to be finite, but those of order at least α to be infinite. It is also strictly stable, since for $\alpha \in (0,1) \cup (1,2)$ it has no drift, and since $\beta = 0$ for $\alpha = 1$. LFSM is in fact a symmetric process for $\alpha = 1$.

The representation for LFSM was introduced by Maejima in [Mae]. LFSM is a collection of processes, since for each fixed α, H, a, and b, $X_{\alpha,H}(a,b)$ appropriately renormalized is usually a different process for different $(a,b) \neq (0,0)$. (See [Sam94, Chapter 7.4] for further details.) The process is *forward anticipating* if $a = 0$, *non-anticipating* if $b = 0$, and *well-balanced* if $a = b$.

The finite-dimensional distributions of LFSM can be characterized using notation from [Ast91]. Define

$$\xi(u) = \begin{cases} |u|^\alpha \left(1 - i\beta\,(\operatorname{sign} u) \tan \frac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1, \\ |u|^\alpha & \text{if } \alpha = 1. \end{cases} \tag{3}$$

with $\operatorname{sign} u = 1, 0$, or -1 if $u > 0, = 0$, or < 0. The characteristic function corresponding to its finite-dimensional distributions is then

$$\mathbb{E} \exp\left\{ i \sum_{i=1}^{d} \theta_j X_{\alpha,H}(a,b;t_j) \right\} = \exp\left\{ -\int_{-\infty}^{\infty} \xi\left(\sum_{i=1}^{d} \theta_j f_{\alpha,H}(a,b;t_j,x) \right) \mathrm{d}x \right\} \tag{4}$$

for $d \geq 1$ and $\theta_j, t_j \in \mathbb{R}, j = 1, \ldots, d$. Since $\beta(x) = \beta$ and equals zero if $\alpha = 1$, then for every $c > 0$ and $t \in \mathbb{R}$,

$$X_{\alpha,H}(a,b;ct) \stackrel{d}{=} c^H X_{\alpha,H}(a,b;t) \tag{5}$$

where $\stackrel{d}{=}$ stands for equality of the finite-dimensional distributions. Relation (5) indicates that LFSM is *H-self-similar* (H-ss). The indices satisfy $0 < \alpha < 2$ and $0 < H < 1, H \neq 1/\alpha$.

The increment process is called the α-stable *linear fractional stable noise.* It is $Y_{\alpha,H}(a,b) = \{Y_{\alpha,H}(a,b;t)\}, t \in \mathbb{R}, |a| + |b| > 0$, with

$$Y_{\alpha,H}(a,b;t) := X_{\alpha,H}(a,b;t+1) - X_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} g_{\alpha,H}(a,b;t,x) M(\mathrm{d}x) \tag{6}$$

and

(7)

$$g_{\alpha,H}(a,b;t,x) := a\left[(t+1-x)_+^{H-\frac{1}{\alpha}} - (t-x)_+^{H-\frac{1}{\alpha}}\right] + b\left[(t+1-x)_-^{H-\frac{1}{\alpha}} - (t-x)_-^{H-\frac{1}{\alpha}}\right],$$

from (1) and (2). $Y_{\alpha,H}(a,b)$ is a *stationary* process since its finite-dimensional distributions are invariant with respect to shifting.

If $\alpha = 2$, LFSM is the *fractional Brownian motion* (FBM), an H-sssi Gaussian process $(0 < H < 1)$ increments. Indeed, FBM is the unique Gaussian H-sssi process. See [**Emb**] or [**Sam94**, Chapter 7] for more study of LFSM and FBM.

It is customary to analyze the stationary increments of any H-sssi process in order to determine the dependence of the process itself. For any stationary process $Y \equiv Y(t), t \in \mathbb{R} = (-\infty, \infty)$, and $\theta_1, \theta_2 \in \mathbb{R}$, the function

(8)

$$r_Y(t) := r_Y(\theta_1, \theta_2; t) = \mathrm{E}\exp\{\mathrm{i}(\theta_1 \mathrm{Y(t)} + \theta_2 \mathrm{Y(0)})\} - \mathrm{E}\exp\{\mathrm{i}\theta_1 \mathrm{Y(t)}\}\mathrm{E}\exp\{\mathrm{i}\theta_2 \mathrm{Y(0)}\}$$

can be used to examine the dependence for the corresponding process of its partial sums. (The function $r(t)$ has been denoted also as $U(t)$: see, for example, [**Ast91**] p. 212.) The case $\theta_1\theta_2 = 0$ is excluded since then $r_Y(t) = 0$.

Previous studies by Astrauskas [**Ast84**] and Astrauskas, Levy, and Taqqu [**Ast91**] apply $r_Y(t) \equiv r(t)$ to $Y \equiv Y_{\alpha,H}(a,b)$. Since LFSN is a moving average, then $r(t)$ converges to 0 as $t \to \infty$. They prove that if

$$0 < \alpha < 2,\ 0 < H < 1,\ H \neq 1/\alpha,\ H \neq 1/[\alpha(\alpha-1)],$$

then the convergence occurs at a hypergeometric rate

$$r(t) \sim c|t|^p,\ p < 0.$$

The coefficient c is specified explicitly and is a complicated function of $\alpha, H, a, b, \theta_1$, and θ_2. The exponent p, which depends only on α and H, is known as the *intensity* of that rate. See Section 2 for precise statements.

The asymptotic dependence structure of any stationary process Y is often determined by the behavior of the series $\sum_{t=-\infty}^{\infty} |r_Y(t)|$. If the series converges then the process is said to display *weak* or *short-range* dependence. It is said to exhibit *long-range* dependence (or has *long memory*) if the series diverges. We mention that this is not the only way to define short-range or long-range dependence; see, for example, [**Sam06**] for other ideas. That structure for LFSN, or for any process for which $r(t)$ behaves hypergeometrically, is controlled primarily by the exponent p: weak dependence occurs if $p < -1$, but long-range dependence occurs if $p \geq -1$.(1)

How fast does $r(t)$ converge to 0 when $1 < \alpha < 2$ at the boundary $H = 1 - 1/[\alpha(\alpha - 1)]$? This is the focus of our paper. After preliminary material is reviewed in Section 2, the main result is given in Section 3. The case of symmetric α-stable ($S\alpha S$) LFSN is included. Section 4 outlines the proofs. The details are carried out in Sections 5 and 6.

The rate of convergence of this measure for some other stationary stable processes besides LFSN can be found, for example, in [**Kok**], [**Lev01**], and [**Lev09**].

2. Preliminaries

We assume that $\theta_1\theta_2 \neq 0$ to avoid $r(t) \equiv 0$. Firstly, for any stationary process Y, rewrite from (8)

(9) $$r_Y(t) = \mathbb{E}e^{i(\theta_1 Y(t)+\theta_2 Y(0))} - \mathbb{E}e^{i(\theta_1 Y(t))}\mathbb{E}e^{i(\theta_2 Y(0))} = e^{-A(\theta_1,\theta_2)}\left(e^{-I_Y(\theta_1,\theta_2;t)} - 1\right)$$

where

(10)
$$I_Y(\theta_1,\theta_2;t) := -\ln \mathbb{E}\exp\{i(\theta_1 Y(t)+\theta_2 Y(0))\} + \ln \mathbb{E}\exp\{i\theta_1 Y(t)\} + \ln \mathbb{E}\exp\{i\theta_2 Y(0)\}.$$

$I_Y(\theta_1,\theta_2;t)$ is the *generalized codifference* of Y since θ_1 and θ_2 are arbitrary. It is evident from (9) that as $|t| \to \infty$

(11) $$r_Y(\theta_1,\theta_2;t) \sim -e^{-A(\theta_1,\theta_2)} I_Y(\theta_1,\theta_2;t) \quad \text{if and only if} \quad I_Y(\theta_1,\theta_2;t) \to 0.$$

The *codifference* is

(12)
$$\tau_Y(t) := -I_Y(1,-1;t) = \ln \mathbb{E}\exp\{i(Y(t)-Y(0))\} - \ln \mathbb{E}\exp\{iY(t)\} - \ln \mathbb{E}\exp\{-iY(0)\}.$$

If Y is symmetric, then

$$\begin{aligned} \tau_Y(t) &= \|Y(t)\|_\alpha^\alpha + \|Y(0)\|_\alpha^\alpha - \|Y(t) - Y(0)\|_\alpha^\alpha \\ &= 2\,\|Y(t)\|_\alpha^\alpha - \|Y(t) - Y(0)\|_\alpha^\alpha \end{aligned}$$

(1) If for LFSN $-1 < p < 0$, the coefficient of asymptoticity vanishes, and the exact asymptotic rate should turn out to be at most $p - 1$, then the process would display weak dependence. However, since the coefficient depends on $\alpha, H, a, b, \theta_1$, and θ_2, it is not easy to establish where it equals zero.