

Central Extension of the Yangian Double

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Abstract

A central extension $\widehat{DY(\mathfrak{g})}$ of the double of the Yangian is defined for a simple Lie algebra $\mathfrak{g}$ with complete proof for $\mathfrak{g} = \mathfrak{sl}_2$. Basic representations and intertwining operators are constructed for $\widehat{DY(\mathfrak{sl}_2)}$.

Résumé

On définit une extension centrale $\widehat{DY(\mathfrak{g})}$ du double du Yangien pour une algèbre de Lie simple $\mathfrak{g}$ avec des preuves complètes pour le cas $\mathfrak{g} = \mathfrak{sl}_2$. On construit des représentations de base et des opérateurs d'entrelacement pour $\widehat{DY(\mathfrak{sl}_2)}$.

1 Introduction

The Yangian $Y(\mathfrak{g})$ was introduced by V. Drinfeld [D1] as a Hopf algebra quantizing the rational solution $r(u) = \frac{e_i \otimes e^i}{u}$ of classical Yang-Baxter equation. As a Hopf algebra $Y(\mathfrak{g})$ is a deformation of the universal enveloping algebra $U(\mathfrak{g}[u])$ of polynomial currents to a simple Lie algebra $\mathfrak{g}$ with respect to cobracket defined by $r(u)$. Unfortunately, up to the moment the representation theory of the Yangian is not so rich in applications as it takes place, for instance, for quantum affine algebras [JM]. We could mention the following gaps:

- (i) The Yangian is not quasitriangular Hopf algebra, but pseudoquasitriangular Hopf algebra [D1];
- (ii) There are no nontrivial examples of infinite-dimensional representations of $Y(\mathfrak{g})$.

In order to get quasitriangular Hopf algebra one should introduce the quantum double $DY(\mathfrak{g})$ of the Yangian. Detailed analysis of $DY(\mathfrak{g})$ was done in [KT] together with explicit description of the universal R -matrix for $DY(\mathfrak{g})$ (complete for $DY(\mathfrak{sl}_2)$ and partial in general case).

The most important examples of infinite dimensional representations of (quantum) affine algebras appear for nonzero value of central charge. Analogously, in the

AMS 1980 *Mathematics Subject Classification* (1985 *Revision*): 17B37, 81R50

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case of the Yangian we could expect the appearance of infinite dimensional representations only after central extension of $DY(\mathfrak{g})$. This program is realized in this paper with complete proof for $DY(sl_2)$: we give a description of central extension $\widehat{DY}(sl_2)$ and construct its basic representations in bosonized form. In general case we present a description of $\widehat{DY}(\mathfrak{g})$ without complete proof. In the forthcoming paper [KLP] we demonstrate, following general scheme of [DFJMN], how our construction produce the formulas for correlation functions in rational models [S].

The central extension of $DY(\mathfrak{g})$ could be constructed in two ways. In Faddev--Reshetikhin-Takhtajan approach [FRT] we could describe $DY(\mathfrak{g})$ by the set of equations

$$(1) \quad R_{V,W}^{\pm}(u-v)L_V^{\pm}(u)L_W^{\pm}(v) = L_W^{\pm}(v)L_V^{\pm}(u)R_{V,W}^{\pm}(u-v),$$

$$(2) \quad R_{V,W}^{+}(u-v)L_V^{+}(u)L_W^{-}(v) = L_W^{-}(v)L_V^{+}(u)R_{V,W}^{+}(u-v)$$

for matrix valued generating functions $L_V^{\pm}(u)$, $L_W^{\pm}(v)$ of $DY(\mathfrak{g})$, where $V(u)$, $W(v)$ are finite dimensional representations of $DY(\mathfrak{g})$, $R_{V,W}^{\pm}$ are images of the universal R -matrix $\mathcal{R}$ and $(\mathcal{R}^{-1})^{21}$. Then, following [RS], we can make a shift of a spectral parameter in R -matrix in equation (2) by a central element:

$$(3) \quad R_{V,W}^{+}(u-v+c\hbar)L_V^{+}(u)L_W^{-}(v) = L_W^{-}(v)L_V^{+}(u)R_{V,W}^{+}(u-v).$$

For the construction of representations we should then extract Drinfeld generators of $\widehat{DY}(\mathfrak{g})$ from L -operators (3) via their Gauss decomposition [DF].

We prefer another way, originally used by V.Drinfeld [D2] for his “current” description of quantum affine algebras. The properties of comultiplication for $Y(\mathfrak{g})$ show that one can extend the Yangian $Y(\mathfrak{g})$ (or its dual with opposite comultiplication $Y^0(\mathfrak{g})$) to a new Hopf algebra, adding the derivative d of a spectral parameter u . Alternatively, one can extend $Y(\mathfrak{g})$ by automorphisms of shifts of u . The double of this extension is exactly what we want to find. Central element c is dual to derivative d .

The plan of the paper is as follows. First we remind the description of the Yangian $Y(sl_2)$ and of its quantum double from [KT]. Then we construct the central extension $\widehat{DY}(sl_2)$, describe the structure of the universal R -matrix for $\widehat{DY}(sl_2)$ and translate our description into L -operator language. In section 5 we construct basic representation of $\widehat{DY}(sl_2)$ and in the last section we describe the structure of $\widehat{DY}(\mathfrak{g})$ in general case (without a proof).

2 $Y(sl_2)$ and its Quantum Double

The Yangian $Y(sl_2)$ can be described as a Hopf algebra generated by the elements $e_k, h_k, f_k, k \geq 0$ subjected to the relations

$$\begin{aligned}
 [h_k, h_l] &= 0, & [e_k, f_l] &= h_{k+l}, \\
 [h_0, e_l] &= 2e_l, & [h_0, f_l] &= -2f_l, \\
 [h_{k+1}, e_l] - [h_k, e_{l+1}] &= \hbar\{h_k, e_l\}, \\
 [h_{k+1}, f_l] - [h_k, f_{l+1}] &= -\hbar\{h_k, f_l\}, \\
 [e_{k+1}, e_l] - [e_k, e_{l+1}] &= \hbar\{e_k, e_l\}, \\
 [f_{k+1}, f_l] - [f_k, f_{l+1}] &= -\hbar\{f_k, f_l\},
 \end{aligned}
 \tag{4}$$

where $\hbar$ is a parameter of the deformation, $\{a, b\} = ab + ba$. The comultiplication and the antipode are uniquely defined by the relations

$$\begin{aligned}
 \Delta(e_0) &= e_0 \otimes 1 + 1 \otimes e_0, & \Delta(h_0) &= h_0 \otimes 1 + 1 \otimes h_0, & \Delta(f_0) &= f_0 \otimes 1 + 1 \otimes f_0, \\
 \Delta(e_1) &= e_1 \otimes 1 + 1 \otimes e_1 + \hbar h_0 \otimes e_0, & \Delta(f_1) &= f_1 \otimes 1 + 1 \otimes f_1 + \hbar f_0 \otimes h_0, \\
 \Delta(h_1) &= h_1 \otimes 1 + 1 \otimes h_1 + \hbar h_0 \otimes h_0 - 2\hbar f_0 \otimes e_0.
 \end{aligned}
 \tag{5}$$

In terms of generating functions

$$e^+(u) := \sum_{k \geq 0} e_k u^{-k-1}, \quad f^+(u) := \sum_{k \geq 0} f_k u^{-k-1}, \quad h^+(u) := 1 + \hbar \sum_{k \geq 0} h_k u^{-k-1}$$

the relations (4) look as follows

$$\begin{aligned}
 [h^+(u), h^+(v)] &= 0, \\
 [e^+(u), f^+(v)] &= -\frac{1}{\hbar} \frac{h^+(u) - h^+(v)}{u - v}, \\
 [h^+(u), e^+(v)] &= -\hbar \frac{\{h^+(u), (e^+(u) - e^+(v))\}}{u - v}, \\
 [h^+(u), f^+(v)] &= \hbar \frac{\{h^+(u), (f^+(u) - f^+(v))\}}{u - v}, \\
 [e^+(u), e^+(v)] &= -\hbar \frac{(e^+(u) - e^+(v))^2}{u - v}, \\
 [f^+(u), f^+(v)] &= \hbar \frac{(f^+(u) - f^+(v))^2}{u - v}.
 \end{aligned}
 \tag{6}$$

The comultiplication is given by Molev's formulas [M], see also [KT]:

$$\Delta(e^+(u)) = e^+(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k \hbar^{2k} (f^+(u + \hbar))^k h^+(u) \otimes (e^+(u))^{k+1},
 \tag{7}$$

$$(8) \quad \Delta(f^+(u)) = 1 \otimes f^+(u) + \sum_{k=0}^{\infty} (-1)^k \hbar^{2k} (f^+(u))^{k+1} \otimes h^+(u) (e^+(u + \hbar))^k$$

$$(9) \quad \Delta(h^+(u)) = \sum_{k=0}^{\infty} (-1)^k (k+1) \hbar^{2k} (f^+(u + \hbar))^k h^+(u) \otimes h^+(u) (e^+(u + \hbar))^k$$

Let now C be an algebra generated by the elements $e_k, f_k, h_k, (k \in \mathbf{Z})$, with relations (4). Algebra C admits $\mathbf{Z}$ -filtration

$$(10) \quad \dots \subset C_{-n} \subset \dots \subset C_{-1} \subset C_0 \subset C_1 \dots \subset C_n \dots \subset C$$

defined by the conditions $\deg e_k = \deg f_k = \deg h_k = k$; $\deg x \in C_m \leq m$. Let $\bar{C}$ be the corresponding formal completion of C . It is proved in [KT] that $DY(sl_2)$ is isomorphic to $\bar{C}$ as an algebra. In terms of generating functions

$$e^{\pm}(u) := \pm \sum_{\substack{k \geq 0 \\ k < 0}} e_k u^{-k-1}, \quad f^{\pm}(u) := \pm \sum_{\substack{k \geq 0 \\ k < 0}} f_k u^{-k-1}, \quad h^{\pm}(u) := 1 \pm \hbar \sum_{\substack{k \geq 0 \\ k < 0}} h_k u^{-k-1},$$

$$e(u) = e^+(u) - e^-(u), \quad f(u) = f^+(u) - f^-(u)$$

the defining relations for $DY(sl_2)$ look as follows:

$$(11) \quad \begin{aligned} h^a(u) h^b(v) &= h^b(v) h^a(u), \quad a, b = \pm, \\ e(u) e(v) &= \frac{u - v + \hbar}{u - v - \hbar} e(v) e(u) \\ f(u) f(v) &= \frac{u - v - \hbar}{u - v + \hbar} f(v) f(u) \\ h^{\pm}(u) e(v) &= \frac{u - v + \hbar}{u - v - \hbar} e(v) h^{\pm}(u) \\ h^{\pm}(u) f(v) &= \frac{u - v - \hbar}{u - v + \hbar} f(v) h^{\pm}(u) \\ [e(u), f(v)] &= \frac{1}{\hbar} \delta(u - v) (h^+(u) - h^-(v)) \end{aligned}$$

Here

$$\delta(u - v) = \sum_{n+m=-1} u^n v^m$$

satisfies the property

$$\delta(u - v) f(u) = \delta(u - v) f(v).$$

The comultiplication is given by the same formulas

$$\begin{aligned}\Delta(e^\pm(u)) &= e^\pm(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k \hbar^{2k} (f^\pm(u + \hbar))^k h^\pm(u) \otimes (e^\pm(u))^{k+1}, \\ \Delta(f^\pm(u)) &= 1 \otimes f^\pm(u) + \sum_{k=0}^{\infty} (-1)^k \hbar^{2k} (f^\pm(u))^{k+1} \otimes h^\pm(u) (e^\pm(u + \hbar))^k \\ (12) \quad \Delta(h^\pm(u)) &= \sum_{k=0}^{\infty} (-1)^k (k+1) \hbar^{2k} (f^\pm(u + \hbar))^k h^\pm(u) \otimes h^\pm(u) (e^\pm(u + \hbar))^k.\end{aligned}$$

Subalgebra $Y^+ = Y(sl_2) \subset DY(sl_2)$ is generated by the components of $e^+(u)$, $f^+(u)$, $h^+(u)$ and its dual with opposite comultiplication $Y^- = (Y(sl_2))^0$ is a formal completion (see (10)) of subalgebra, generated by the components of $e^-(u)$, $f^-(u)$, $h^-(u)$.

Let us describe the Hopf pairing of Y^+ and Y^- [KT]. Note that by a Hopf pairing $\langle, \rangle: A \otimes B \rightarrow \mathbf{C}$ we mean bilinear map satisfying the conditions

$$(13) \quad \langle a, b_1 b_2 \rangle = \langle \Delta(a), b_1 \otimes b_2 \rangle, \quad \langle a_1 a_2, b \rangle = \langle a_2 \otimes a_1, \Delta(b) \rangle$$

The last condition is unusual but convenient in a work with quantum double.

Let $E^\pm$, $H^\pm$, $F^\pm$ be subalgebras (or their completions in Y^- case) generated by the components of $e^\pm(u)$, $h^\pm(u)$, $f^\pm(u)$. Subalgebras $E^\pm$ and $F^\pm$ do not contain the unit.

The first property of the Hopf pairing $Y^+ \otimes Y^- \rightarrow \mathbf{C}$ is that it preserves the decompositions

$$Y^+ = E^+ H^+ F^+, \quad Y^- = F^- H^- E^-.$$

It means that

$$(14) \quad \langle e^+ h^+ f^+, f^- h^- e^- \rangle = \langle e^+, f^- \rangle \langle h^+, h^- \rangle \langle f^+, e^- \rangle$$

for any elements $e^\pm \in E^\pm$, $h^\pm \in H^\pm$, $f^\pm \in F^\pm$. This property defines the pairing uniquely together with the relations

$$\begin{aligned}\langle e^+(u), f^-(v) \rangle &= \frac{1}{\hbar(u-v)}, & \langle f^+(u), e^-(v) \rangle &= \frac{1}{\hbar(u-v)}, \\ \langle h^+(u), h^-(v) \rangle &= \frac{u-v+\hbar}{u-v-\hbar}.\end{aligned}$$

The full information of the pairing is encoded in the universal R -matrix for $DY(sl_2)$ which has the following form [KT]:

$$(15) \quad \mathcal{R} = R_+ R_0 R_- ,$$