

ANALYTIC CONTINUATION IN REPRESENTATION THEORY AND HARMONIC ANALYSIS

by

Gestur Ólafsson

Abstract. — In this paper we discuss topics in harmonic analysis and representation theory related to two different real forms G/H and G^c/H of a complex semisimple symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$. We connect representations of G and G^c using the theory of involutive representations of semi-groups and reflection symmetry. We discuss how to generalize the Segal-Bargmann transform to real forms of bounded symmetric domains. This transform maps $L^2(H/H \cap K)$ into the representation space of a highest weight representation of G . We show how this transform is related to reflection symmetry, which shows that it is a natural transform related to representation theory. Finally we discuss the connection of the H -spherical characters of the representations and relate them to spherical functions.

Résumé (Prolongement analytique en théorie des représentations et analyse harmonique)

Dans cet article, nous considérons des questions en analyse harmonique et en théorie des représentations concernant deux formes réelles différentes G/H et G^c/H d'un espace symétrique semi-simple complexe $G_{\mathbb{C}}/H_{\mathbb{C}}$. Nous établissons un lien entre les représentations de G et de G^c à l'aide de la théorie des représentations involutives des semi-groupes et la symétrie de réflexion. On examine la question de la généralisation de la transformée de Segal-Bargmann aux formes réelles des domaines symétriques bornés. Cette transformée envoie l'espace $L^2(H/H \cap K)$ dans l'espace de représentations d'une représentation du poids maximum de G . Nous montrons comment cette transformée est liée à la symétrie de réflexion, ce qui montre que c'est une transformée naturelle liée à la théorie des représentations. Finalement, on étudie la relation entre les caractères H -sphériques des représentations et les fonctions sphériques.

1. Introduction

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $G_{\mathbb{C}}$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We will for simplicity assume that $G \subset G_{\mathbb{C}}$ even if most of what we say holds also for the universal covering

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group $\tilde{G}$ and other connected groups locally isomorphic to G . Let $\theta : G \rightarrow G$ a Cartan involution on G and denote by $K = G^\theta$ the corresponding maximal compact subgroup of G . We will be interested in a special class of symmetric spaces, that are closely related to real forms of *bounded symmetric domains*. We will therefore assume that $D = G/K$ is a bounded symmetric domain. Let $\tau : D \rightarrow D$ by a *conjugation*, i.e., an anti-holomorphic involution, fixing the point $\{K\} \in D$. Those involutions were classified by A. Jaffee in [25, 26]. We will give the list later. We lift τ to an involution on G , $G_{\mathbb{C}}$, $\mathfrak{g}$, and $\mathfrak{g}_{\mathbb{C}}$. We will also denote those involutions by τ . Then τ commutes with θ . Let $H = G^\tau \subset H_{\mathbb{C}} := G_{\mathbb{C}}^\tau$. Then $D^\tau = H/H \cap K$.

On the Lie algebra level we have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{q} \\ &= \mathfrak{k} \oplus \mathfrak{p} \\ &= (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}), \end{aligned}$$

$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\}$, $\mathfrak{h} = \{X \in \mathfrak{g} \mid \tau X = X\}$, and $\mathfrak{q} = \{X \in \mathfrak{g} \mid \tau X = -X\}$. Define a new real form of $\mathfrak{g}_{\mathbb{C}}$ by

$$\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$$

and let G^c be the corresponding real analytic subgroup of $G_{\mathbb{C}}$. Denote also by τ the restriction of τ to G^c . Let $H^c := G^{c\tau}$. Then $H = H^c = G \cap G^c$.

We have the following diagrams

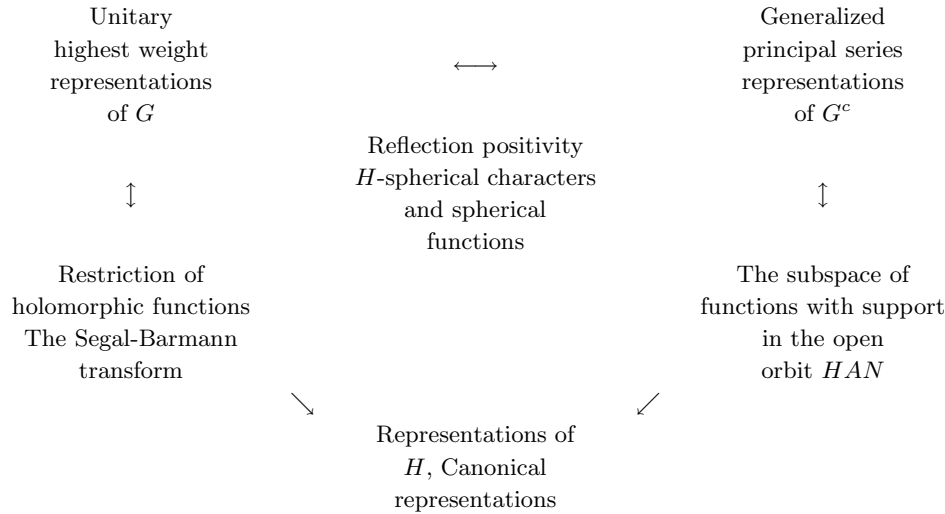
$$\begin{array}{ccc} & M_{\mathbb{C}} := G_{\mathbb{C}}/H_{\mathbb{C}} & \\ & \text{Complex} & \\ M := G/H & \nearrow & \nwarrow M^c := G^c/H \\ & \text{Real forms} & \end{array}$$

and

$$D^\tau = H/H \cap K \xrightarrow[\text{Real form}]{} D = G/K.$$

The ideas that we discuss here are *how to analyze representations of G , G^c , and H via analytic continuation to open domains in $M_{\mathbb{C}}$ or by restriction to a real form*. The main tools are *involutive representations and positive definite kernels*. This can be

expressed by the following simple diagram:



Most of the ideas discussed here have been explained before in [30, 31, 49, 51]. We would in particular like to point to [31] for discussion on reflection positivity and highest weight representations. Several other people have been working on similar projects. We would like to point out here the following papers and preprints [1, 2, 7, 8, 19, 41, 43, 56, 64, 68].

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2. Unitary highest weight representations

We use the same notation and assumptions as in the introduction. In particular we assume that $G \subset G_{\mathbb{C}}$ is a Hermitian group, and $G_{\mathbb{C}}$ is simply connected. Thus $D = G/K$ is a bounded symmetric domain. The complex structure on D corresponds to an element $Z^0 \in \mathfrak{z}(\mathfrak{k})$ such that $\text{ad}(Z^0)$ has eigenvalues $0, i, -i$. The eigenspace

corresponding to 0 is $\mathfrak{k}_{\mathbb{C}}$, and we denote by $\mathfrak{p}^+$, respectively $\mathfrak{p}^-$ the eigenspace corresponding to i respectively $-i$. Then both $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subalgebras of $\mathfrak{g}_{\mathbb{C}}$ and

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-.$$

Let $K_{\mathbb{C}} := \exp(\mathfrak{k}_{\mathbb{C}})$, and $P^{\pm} := \exp(\mathfrak{p}^{\pm})$. The restriction of the exponential map is an isomorphism of $\mathfrak{p}^{\pm}$ onto $P^{\pm}$. The set $P^+ K_{\mathbb{C}} P^-$ is open and dense in $G_{\mathbb{C}}$. Furthermore multiplication induces a diffeomorphism

$$P^+ \times K_{\mathbb{C}} \times P^- \ni (p, k, q) \mapsto pkq \in P^+ K_{\mathbb{C}} P^- \subset G_{\mathbb{C}}.$$

We denote the inverse map by $x \mapsto (p(x), k(x), q(x))$. The Harish-Chandra bounded realization of D is given by

$$(2.1) \quad G/K \ni gK \mapsto \log(p(g)) \in \Omega_{\mathbb{C}} \subset \mathfrak{p}^+$$

and $\Omega_{\mathbb{C}}$ is a bounded symmetric domain in $\mathfrak{p}^+$. Let $(\pi, \mathbf{H})$ be a representation of G in a Hausdorff, locally convex complete topological vector space $\mathbf{H}$, and let L be a closed subgroup of G . A vector $\mathbf{v} \in \mathbf{H}$ is called L -finite if $\pi(L)\mathbf{v}$ spans a finite dimensional subspace of $\mathbf{H}$. We call $\mathbf{v}$ smooth or analytic if for all $X \in \mathfrak{g}$ the map

$$\mathbb{R} \ni t \mapsto \pi(\exp(tX))\mathbf{v} \in \mathbf{H}$$

is smooth, or analytic respectively. We denote by $\mathbf{H}_L$, $\mathbf{H}^{\infty}$, $\mathbf{H}^{\omega}$ the space of L -finite, smooth, respectively analytic vectors in $\mathbf{H}$. Define a representation of $\mathfrak{g}$ on $\mathbf{H}^{\infty}$ by

$$d\pi(X)\mathbf{u} := \lim_{t \rightarrow 0} \frac{\pi(\exp tX)\mathbf{u} - \mathbf{u}}{t}, \quad \mathbf{u} \in \mathbf{H}^{\infty}.$$

We extend $d\pi$ by linearity to a representation of $\mathfrak{g}_{\mathbb{C}}$ and then of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. The representations of G that we are mainly interested in are the *unitary highest weight representations* of G (see [5, 6, 9, 10, 15, 16, 22, 27, 42, 61, 65, 67] for further information.) Let $(\pi, \mathbf{H})$ be an admissible representation of G in a Banach space, and assume that the center of $U(\mathfrak{g})$ acts by scalars. Then $\mathbf{H}_K \subset \mathbf{H}^{\omega}$ and $\mathbf{H}_K$ is an $(U(\mathfrak{g}), K)$ -module in the sense that it is both an $U(\mathfrak{g})$ and a K -module such that

$$X \cdot (k \cdot \mathbf{u}) = (\text{Ad}(k)X) \cdot \mathbf{u}, \quad \forall k \in K, X \in \mathfrak{g}, \mathbf{u} \in \mathbf{H}_K.$$

We say that an $(U(\mathfrak{g}), K)$ -module $(\pi, \mathbf{H})$ is *admissible* if the multiplicity of each irreducible representation of K in $\mathbf{H}$ is finite. Let $\mathfrak{t} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$ containing Z^0 . Then $\mathfrak{t} \subset \mathfrak{z}_{\mathfrak{g}}(Z^0) \subset \mathfrak{k}$ so $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$.

Definition 2.1. — *Let $\mathbf{H}$ be an $(U(\mathfrak{g}), K)$ -module. Then π is called a highest weight representation if there exists a Borel subalgebra $\mathfrak{p} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$, $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, and $\mathbf{v} \in \mathbf{H}$ such that the following holds:*

1. $X \cdot \mathbf{v} = \lambda(X)\mathbf{v}$ for all $X \in \mathfrak{t}$;
2. $\pi(\mathfrak{u})\mathbf{v} = 0$;
3. $U(\mathfrak{g})\mathbf{v} = \mathbf{H}$.

The element $\mathbf{v}$ is called primitive element of weight λ .

All the irreducible unitary highest weight representations of G can be constructed in a space of holomorphic functions on D for an appropriate choice of Z^0 or, which is the same, complex structure on D . For that let π be an irreducible representation of K . Let Δ denote the set of roots of $\mathfrak{k}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. A root α is called *compact* if $\mathfrak{g}_{\mathbb{C}\alpha} \subset \mathfrak{k}_{\mathbb{C}}$. Otherwise α is called *non-compact*. Let Δ_c respectively Δ_n be the set of compact respectively non-compact roots. We choose the set Δ^+ of positive roots such that $\Delta_c^+ := \Delta^+ \cap \Delta_c$ is a system of positive roots for Δ_c and $\Delta_n^+ := \Delta^+ \cap \Delta_n = \{\alpha \in \Delta \mid \mathfrak{g}_{\mathbb{C}\alpha} \subset \mathfrak{p}^+\}$. Choose $H_\alpha \in i\mathfrak{t}$, such that $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(H_\alpha) = 2$. Let $W_K = W(\Delta_c) \subset W = W(\Delta)$ be the Weyl group generated by the reflections $s_\alpha(X) = X - \alpha(X)H_\alpha$ for $\alpha \in \Delta_c$ respectively $\alpha \in \Delta$. We denote the corresponding reflection on $i\mathfrak{t}^*$ by the same letter, i.e. $s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$. Then Δ_n^+ is invariant under W_K . Let $\sigma : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ be the conjugation with respect to G . We sometimes write W or $\bar{g}$ for $\sigma(W)$, respectively $\sigma(g)$. We will usually use capital letters for the elements of the Lie algebra $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ except where we are viewing them as complex variables or elements in $\Omega_{\mathbb{C}}$.

Let $(\pi, \mathbf{V})$ be an irreducible representation of K with highest weight $\mu = \mu(\pi) \in i\mathfrak{t}^*$. For $z, v, w \in \mathfrak{p}^+$ and $g \in G_{\mathbb{C}}$ such that $g \exp(z), \exp(-w) \exp(v) \in P^+ K_{\mathbb{C}} P^-$, let

$$(2.2) \quad g \cdot z := \log(p(g \exp z))$$

$$(2.3) \quad J(g, z) := k(g \exp z), \text{ and}$$

$$(2.4) \quad \kappa(v, w) := k(\exp(-\bar{w}) \exp(v)).$$

Then the isomorphism in (2.1) intertwines the natural G -action on D with the action $(g, z) \mapsto g \cdot z$ of G on $\Omega_{\mathbb{C}}$. The function $J(g, z)$ is called *the universal factor of automorphy*. Finally we define

$$(2.5) \quad J_\pi(g, z) := \pi(J(g, z)) \quad \text{and} \quad K_\pi(z, w) := \pi(\kappa(z, w))^{-1}.$$

Then $z, w \mapsto K_\pi(z, w)$ is holomorphic in the first variable and anti-holomorphic in the second variable.

Let $S(\Omega_{\mathbb{C}}) := \{g \in G_{\mathbb{C}} \mid g^{-1} \cdot \Omega_{\mathbb{C}} \subset \Omega_{\mathbb{C}}\}$. Then $S(\Omega_{\mathbb{C}})$ is a maximal closed semi-group in $G_{\mathbb{C}}$ and there exists a maximal closed and convex G -invariant cone $W_{\max} \subset \mathfrak{g}$ such that $W_{\max} \cap -W_{\max} = \{0\}$ (pointed), $W_{\max} - W_{\max} = \mathfrak{g}$ (generating), and (see [21, 23, 44])

$$S(\Omega_{\mathbb{C}}) = G \exp(iW_{\max}).$$

Let $W_{\min} = \{X \in \mathfrak{g} \mid \forall Y \in W_{\max} : -B(X, \theta(Y)) \geq 0\}$, where B stands for the Killing form. Then $W_{\min}$ is a minimal G -invariant pointed and generating cone in $\mathfrak{g}$. Define $g^* = \sigma(g)^{-1}$. If $s = g \exp(iW) \in S(\Omega_{\mathbb{C}})$ then

$$s^* = \exp(iW)g^{-1} = g^{-1} \exp(i\text{Ad}(g)W) \in S(\Omega_{\mathbb{C}}).$$

We notice the following well known and important lemma.