

RESOLVING 3-DIMENSIONAL TORIC SINGULARITIES

by

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Abstract. — This paper surveys, in the first place, some basic facts from the classification theory of normal complex singularities, including details for the low dimensions 2 and 3. Next, it describes how the toric singularities are located within the class of rational singularities, and recalls their main properties. Finally, it focuses, in particular, on a toric version of Reid’s desingularization strategy in dimension three.

1. Introduction

There are certain general *qualitative criteria* available for the rough classification of singularities of complex varieties. The main ones arise:

- from the study of the punctual algebraic behaviour of these varieties
(w.r.t. local rings associated to singular points)
[*algebraic classification*]
- from an intrinsic characterization
for the nature of the possible exceptional loci w.r.t. any desingularization
[*rational, elliptic, non-elliptic etc.*]
- from the behaviour of “discrepancies”
(for $\mathbb{Q}$ -Gorenstein normal complex varieties)
[*adjunction-theoretic classification*]

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Algebraic Classification. — At first we recall some fundamental definitions from commutative algebra (cf. [52], [54]). Let R be a commutative ring with 1. The *height* $\text{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of R is the supremum of the lengths of all prime ideal chains which are contained in $\mathfrak{p}$, and the *dimension* of R is defined to be

$$\dim(R) := \sup \{ \text{ht}(\mathfrak{p}) \mid \mathfrak{p} \text{ prime ideal of } R \}.$$

R is *Noetherian* if any ideal of it has a finite system of generators. R is a *local ring* if it is endowed with a *unique* maximal ideal $\mathfrak{m}$. A local ring R is *regular* (resp. *normal*) if $\dim(R) = \dim(\mathfrak{m}/\mathfrak{m}^2)$ (resp. if it is an integral domain and is integrally closed in its field of fractions). A finite sequence $a_1, \dots, a_\nu$ of elements of a ring R is defined to be a *regular sequence* if a_1 is not a zero-divisor in R and for all i , $i = 2, \dots, \nu$, a_i is not a zero-divisor of $R/\langle a_1, \dots, a_{i-1} \rangle$. A Noetherian local ring R (with maximal ideal $\mathfrak{m}$) is called *Cohen-Macaulay* if

$$\text{depth}(R) = \dim(R),$$

where the *depth* of R is defined to be the maximum of the lengths of all regular sequences whose members belong to $\mathfrak{m}$. A Cohen-Macaulay local ring R is called *Gorenstein* if

$$\text{Ext}_R^{\dim(R)}(R/\mathfrak{m}, R) \cong R/\mathfrak{m}.$$

A Noetherian local ring R is said to be a *complete intersection* if there exists a regular local ring R' , such that $R \cong R'/(f_1, \dots, f_q)$ for a finite set $\{f_1, \dots, f_q\} \subset R'$ whose cardinality equals $q = \dim(R') - \dim(R)$. The hierarchy by inclusion of the above types of Noetherian local rings is known to be described by the following diagram:

$$(1.1) \quad \begin{array}{ccc} \{\text{Noetherian local rings}\} & \supset & \{\text{normal local rings}\} \\ \cup & & \cup \\ \{\text{Cohen-Macaulay local rings}\} & & \{\text{regular local rings}\} \\ \cup & & \cap \\ \{\text{Gorenstein local rings}\} & \supset & \{\text{complete intersections ("c.i.'s")}\} \end{array}$$

An arbitrary Noetherian ring R and its associated affine scheme $\text{Spec}(R)$ are called Cohen-Macaulay, Gorenstein, normal or regular, respectively, iff all the localizations $R_{\mathfrak{m}}$ with respect to all the members $\mathfrak{m} \in \text{Max-Spec}(R)$ of the maximal spectrum of R are of this type. In particular, if the $R_{\mathfrak{m}}$'s for all maximal ideals $\mathfrak{m}$ of R are c.i.'s, then one often says that R is a *locally complete intersection* ("l.c.i.") to distinguish it from the "global" ones. (A *global complete intersection* ("g.c.i.") is defined to be a ring R of finite type over a field $\mathbf{k}$ (i.e., an affine $\mathbf{k}$ -algebra), such that

$$R \cong \mathbf{k}[T_1, \dots, T_d] / (\varphi_1(T_1, \dots, T_d), \dots, \varphi_q(T_1, \dots, T_d))$$

for q polynomials $\varphi_1, \dots, \varphi_q$ from $\mathbf{k}[T_1, \dots, T_d]$ with $q = d - \dim(R)$). Hence, the above inclusion hierarchy can be generalized for all Noetherian rings, just by omitting in (1.1) the word "local" and by substituting l.c.i.'s for c.i.'s.

We shall henceforth consider only *complex varieties* $(X, \mathcal{O}_X)$, i.e., integral separated schemes of finite type over $k = \mathbb{C}$; thus, the punctual algebraic behaviour of X is determined by the stalks $\mathcal{O}_{X,x}$ of its structure sheaf $\mathcal{O}_X$, and X itself is said to have a given *algebraic property* whenever all $\mathcal{O}_{X,x}$'s have the analogous property from (1.1) for all $x \in X$. Furthermore, via the GAGA-correspondence ([71], [30, § 2]) which preserves the above quoted algebraic properties, we may work within the *analytic category* by using the usual contravariant functor

$$(X, x) \rightsquigarrow \mathcal{O}_{X,x}^{\text{hol}}$$

between the category of isomorphy classes of germs of X and the corresponding category of isomorphy classes of analytic local rings at the marked points x . For a complex variety X and $x \in X$, we denote by $\mathfrak{m}_{X,x}$ the maximal ideal of $\mathcal{O}_{X,x}^{\text{hol}}$ and by

$$(1.2) \quad \begin{aligned} \text{Sing}(X) &= \{x \in X \mid \mathcal{O}_{X,x}^{\text{hol}} \text{ is a non-regular local ring}\} \\ &= \{x \in X \mid \dim(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2) > \dim_x(X)\} \end{aligned}$$

its *singular locus*. By a *desingularization* (or *resolution of singularities*) $f : \widehat{X} \rightarrow X$ of a non-smooth X , we mean a “full” or “overall” desingularization (if not mentioned), i.e., $\text{Sing}(\widehat{X}) = \emptyset$. When we deal with *partial* desingularizations, we mention it explicitly.

Rational and Elliptic Singularities. — We say that X has (at most) *rational singularities* if there exists a desingularization $f : Y \rightarrow X$ of X , such that

$$f_* \mathcal{O}_Y = \mathcal{O}_X$$

(equivalently, Y is normal), and

$$R^i f_* \mathcal{O}_Y = 0, \quad \forall i, \quad 1 \leq i \leq \dim_{\mathbb{C}} X - 1.$$

(The i -th direct image sheaf is defined via

$$U \longmapsto R^i f_* \mathcal{O}_Y(U) := H^i(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)}).$$

This definition is independent of the particular choice of the desingularization of X . (Standard example: quotient singularities⁽¹⁾ are rational singularities).

We say that a Gorenstein singularity x of X is an *elliptic singularity* if there exists a desingularization $f : Y \rightarrow X$ of $x \in X$, such that

$$R^i f_* \mathcal{O}_Y = 0, \quad \forall i, \quad 1 \leq i \leq \dim_{\mathbb{C}} X - 2,$$

⁽¹⁾The *quotient singularities* are of the form $(\mathbb{C}^r/G, [\mathbf{0}])$, where G is a finite subgroup of $\text{GL}(r, \mathbb{C})$ (without pseudoreflections) acting linearly on $\mathbb{C}^r$, $p : \mathbb{C}^r \rightarrow \mathbb{C}^r/G = \text{Spec}(\mathbb{C}[\mathbf{z}]^G)$ the quotient map, and $[\mathbf{0}] = p(\mathbf{0})$. Note that

$$\text{Sing}(\mathbb{C}^r/G) = p(\{z \in \mathbb{C}^r \mid Gz \neq \{\text{Id}\}\})$$

(cf. (1.2)), where $G_z := \{g \in G \mid g \cdot z = z\}$ is the isotropy group of $z \in \mathbb{C}^r$.

and

$$R^{\dim X - 1} f_* \mathcal{O}_Y \cong \mathbb{C}.$$

(The definition is again independent of the particular choice of the desingularization).

Adjunction-Theoretic Classification. — If X is a *normal* complex variety, then its Weil divisors can be described by means of “divisorial” sheaves as follows:

Lemma 1.1 ([34, 1.6]). — *For a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ -modules the following conditions are equivalent:*

- (i) $\mathcal{F}$ is reflexive (i.e., $\mathcal{F} \cong \mathcal{F}^{\vee\vee}$, with $\mathcal{F}^\vee := \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ denoting the dual of $\mathcal{F}$) and has rank one.
- (ii) If X^0 is a non-singular open subvariety of X with $\operatorname{codim}_X(X \setminus X^0) \geq 2$, then $\mathcal{F}|_{X^0}$ is invertible and

$$\mathcal{F} \cong \iota_* (\mathcal{F}|_{X^0}) \cong \iota_* \iota^* (\mathcal{F}),$$

where $\iota : X^0 \hookrightarrow X$ denotes the inclusion map.

The *divisorial sheaves* are exactly those satisfying one of the above conditions. Since a divisorial sheaf is torsion free, there is a non-zero section $\gamma \in H^0(X, \operatorname{Rat}_X \otimes_{\mathcal{O}_X} \mathcal{F})$, with

$$H^0(X, \operatorname{Rat}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \cong \mathbb{C}(X) \cdot \gamma,$$

and $\mathcal{F}$ can be considered as a subsheaf of the constant sheaf Rat_X of rational functions of X , i.e., as a special *fractional ideal sheaf*.

Proposition 1.2 ([63, App. of § 1]). — *The correspondence*

$$\operatorname{Cl}(X) \ni \{D\} \xrightarrow{\delta} \{\mathcal{O}_X(D)\} \in \left\{ \begin{array}{l} \text{divisorial coherent} \\ \text{subsheaves of } \operatorname{Rat}_X \end{array} \right\} / H^0(X, \mathcal{O}_X^*)$$

with $\mathcal{O}_X(D)$ defined by sending every non-empty open set U of X onto

$$U \longmapsto \mathcal{O}_X(D)(U) := \{\varphi \in \mathbb{C}(X)^* \mid (\operatorname{div}(\varphi) + D)|_U \geq 0\} \cup \{0\},$$

is a bijection, and induces a $\mathbb{Z}$ -module isomorphism. In fact, to avoid torsion, one defines this $\mathbb{Z}$ -module structure by setting

$$\delta(D_1 + D_2) := (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee} \text{ and } \delta(\kappa D) := \mathcal{O}_X(D)^{[\kappa]} = \mathcal{O}_X(\kappa D)^{\vee\vee},$$

for any Weil divisors D, D_1, D_2 and $\kappa \in \mathbb{Z}$.

Let now $\Omega_{\operatorname{Reg}(X)/\mathbb{C}}$ be the sheaf of regular 1-forms, or Kähler differentials, on

$$\operatorname{Reg}(X) = X \setminus \operatorname{Sing}(X) \xrightarrow{\iota} X,$$

(cf. [36, § 5.3]) and for $i \geq 1$, let us set

$$\Omega_{\operatorname{Reg}(X)/\mathbb{C}}^i := \bigwedge^i \Omega_{\operatorname{Reg}(X)/\mathbb{C}}.$$

The unique (up to rational equivalence) Weil divisor K_X , which maps under δ to the *canonical divisorial sheaf*

$$\omega_X := \iota_* \left(\Omega_{\text{Reg}(X)/\mathbb{C}}^{\dim_{\mathbb{C}}(X)} \right),$$

is called the *canonical divisor* of X . Another equivalent interpretation of ω_X , when X is Cohen-Macaulay, can be given by means of the Duality Theory (see [32], [29]). If $\mathbb{D}_c^+(\mathcal{O}_X)$ denotes the derived category of below bounded complexes whose cohomology sheaves are coherent, then there exists a *dualizing complex*⁽²⁾ $\omega_X^\bullet \in \mathbb{D}_c^+(\mathcal{O}_X)$ over X . If X is Cohen-Macaulay, then the i -th cohomology sheaf $\mathcal{H}^i(\omega_X^\bullet)$ vanishes for all $i \in \mathbb{Z} \setminus \{-\dim_{\mathbb{C}}(X)\}$, and $\omega_X \cong \mathcal{H}^{-\dim_{\mathbb{C}}(X)}(\omega_X^\bullet)$. This leads to the following:

Proposition 1.3. — *A normal complex variety X is Gorenstein if and only if it is Cohen-Macaulay and ω_X is invertible.*

Proof. — If X is Gorenstein, then $\mathcal{O}_{X,x}$ satisfies the equivalent conditions of [54, Thm.18.1], for all $x \in X$. This means that $\mathcal{O}_{X,x}$ (as Noetherian local ring) is a dualizing complex for itself (cf. [32, Ch. V, Thm.9.1, p.293]). Since dualizing complexes are unique up to tensoring with an invertible sheaf, say L , over X , shifted by an integer n (cf. [32, Ch. V, Cor.2.3, p.259]), we shall have $\omega_X^\bullet \cong \mathcal{O}_{X,x}^\bullet \otimes L[n]$. Hence, ω_X itself will be also invertible. The converse follows from the isomorphisms $\omega_X \cong \mathcal{H}^{-\dim_{\mathbb{C}}(X)}(\omega_X^\bullet)$ and $\omega_{X,x} \cong \mathcal{O}_{X,x}$, for all $x \in X$. (Alternatively, one may use the fact that $x \in X$ is Gorenstein iff $\mathcal{O}_{X,x}$ is Cohen-Macaulay and $H_{\mathfrak{m}_{X,x}}^{\dim_{\mathbb{C}}(X)}(\mathcal{O}_{X,x})$ is a dualizing module for it, cf. [29, Prop.4.14, p.65]. The classical duality [29, Thm.6.3, p.85], [32, Ch. V, Cor.6.5, p.280], combined with the above uniqueness argument, gives again the required equivalence). $\square$

Theorem 1.4 (Kempf [43, p.50], Elkik [23], [24], Bingener-Storch [5])

Let X a normal complex variety of dimension ≥ 2 . Then

$$\left(\begin{array}{c} X \text{ has at most} \\ \text{rational singularities} \end{array} \right) \iff \left(\begin{array}{c} X \text{ is Cohen-Macaulay} \\ \text{and } \omega_X \cong f_*\omega_Y \end{array} \right),$$

where $f : Y \rightarrow X$ is any desingularization of X .

(Note that, if $E = f^{-1}(\text{Sing}(X))$ and $\iota : \text{Reg}(X) \hookrightarrow X$, $j : Y \setminus E \hookrightarrow Y$ are the natural inclusions, then by the commutative diagram

$$\begin{array}{ccc} Y \setminus E & \xhookrightarrow{j} & Y \\ \downarrow \iota & & \downarrow f \\ \text{Reg}(X) & \xhookrightarrow{\iota} & X \end{array}$$

⁽²⁾There is a canonical morphism $\omega_X[\dim_{\mathbb{C}}(X)] \rightarrow \omega_X^\bullet$ which is a quasi-isomorphism iff X is Cohen-Macaulay.