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ON THE ASYMPTOTICS OF GREEN'S FUNCTIONS OF ELLIPTIC OPERATORS WITH CONSTANT COEFFICIENTS

by

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***Abstract.* —** In this paper we discuss the following problem. Given an elliptic operator $P(D)$ with constant coefficients in $\mathbb{R}^n$ ($P(\xi) \neq 0$ in $\mathbb{R}^n$) and an infinite cone Γ in $\mathbb{R}^n$, give conditions which ensure that the corresponding Green's function $G(x)$ admits a nice asymptotic behavior as $|x| \to \infty$ in Γ. A solution to the problem is presented and some concrete applications are given. These are related to results by Evgrafov and Postnikov.

***Résumé* (Sur le comportement asymptotique des fonctions de Green des opérateurs elliptiques à coefficients constants)**

Dans cet article nous considérons le problème suivant. Étant donné un opérateur elliptique à coefficients constants, $P(D)$, dans $\mathbb{R}^n$ ($P(\xi) \neq 0$ dans $\mathbb{R}^n$), et un cône infini Γ dans $\mathbb{R}^n$, quelles sont les conditions pour que la fonction de Green associée $G(x)$ ait un bon comportement asymptotique lorsque $|x| \to \infty$ dans Γ ? Nous présentons une solution à ce problème ainsi que des applications. Ceci est relié à des travaux de Evgrafov et Postnikov.

1. Introduction

Let $P(D)$ be an elliptic operator with complex constant coefficients, of even order m, acting on functions on $\mathbb{R}^n$ ($D = (D_1, \cdots, D_n)$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$). Suppose that the polynomial $P(\xi) \neq 0$ for $\xi \in \mathbb{R}^n$. The Green's function $G(x)$ of $P(D)$ on $\mathbb{R}^n$ is given by

$$G(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x}}{P(\xi)} d\xi, \quad x \in \mathbb{R}^n \smallsetminus \{0\} \tag{1.1}$$

where the integral is understood in the distribution sense.

As is well known $G(x)$ is a smooth function on $\mathbb{R}^n \smallsetminus \{0\}$ with a singularity at $x = 0$. $G(x)$ decays exponentially as $|x| \to \infty$.

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In this paper we propose to characterize a class of elliptic operators $P(D)$, $P(\xi) \neq 0$ on $\mathbb{R}^n$, possessing a Green's function with a nice asymptotic behavior as $|x| \to \infty$ ($x \in \mathbb{R}^n$ or, more generally, $x \in \Gamma$ where Γ is some infinite cone in $\mathbb{R}^n$). A prototype of such operators is the Helmholtz operator: $P = -\Delta - \lambda$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$ whose Green's function $G_\lambda(x)$ has the following well known asymptotic formula (derived classically from the asymptotic formula for the Bessel functions). For $0 < \pm \arg \lambda \leqslant \pi$:

$$G_\lambda(x) = c_\pm \lambda^{(n-3)/4} |x|^{-(n-1)/2} e^{\pm i \lambda^{1/2} |x|} \left(1 + O\left(1/|x|\right)\right) \tag{1.2}$$

as $|x| \to \infty$ where $c_\pm = \frac{1}{2}(2\pi)^{-(n-1)/2} e^{\mp i\pi(n-3)/2}$. (Formula (1.2) is also valid for $G_{\lambda \pm i0}(x)$, $\lambda > 0$).

We mention some known results on asymptotic behavior of Green's functions of higher order elliptic operators. First we mention the following results which apply to a class of elliptic operators with constant coefficients different from the class of operators we study here. Suppose that $P(D)$ is positively elliptic: $P(\xi)$ is real for $\xi \in \mathbb{R}^n$, $P(\xi) > 0$ for large $|\xi|$. Suppose further that the set: $M = \{\xi \in \mathbb{R}^n : P(\xi) = 0\}$ is a non-empty connected C^∞ manifold, $P'(\xi) \neq 0$ on M. In this case there are two distinguished Green's functions defined by

$$G_\pm(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x}}{P(\xi) \pm i0} d\xi. \tag{1.3}$$

If the manifold M is strictly convex it was shown by Vainberg [**5**] that the Green's functions $G_\pm(x)$ possess asymptotic formulas of the form:

$$G_\pm(x) = a_\pm(x) e^{\pm i K(x)} \left(1 + O\left(1/|x|\right)\right) \tag{1.4}$$

as $|x| \to \infty$ where $K(x)$ is some real, smooth, convex homogeneous function of degree 1 and $a_\pm(x)$ are certain smooth nowhere zero homogeneous functions of degree $-(n-1)/2$ on $\mathbb{R}^n \smallsetminus \{0\}$ (K and $a_\pm$ admit explicit expressions in terms of the manifold M).

For higher order elliptic operators $P(D)$ such that $P(\xi) \neq 0$ on $\mathbb{R}^n$ (the class of operators which interests us here) an asymptotic formula for the Green's function was established by Evgrafov and Postnikov [**1**] for a rather special class of operators. The main result in [**1**], for the elliptic Green's function, can be formulated as follows.

Theorem 1.1. — *Let $P_0(D)$ be an elliptic operator on $\mathbb{R}^n$. Suppose that the form $P_0(\xi)$ is a positive homogeneous polynomial of even degree m on $\mathbb{R}^n \smallsetminus \{0\}$. Write $P_0(\xi)$ in the form:*

$$P_0(\xi) = \sum_{|\alpha|=m} a_\alpha \binom{m}{\alpha} \xi^\alpha.$$

Suppose that $P_0(\xi)$ verifies the following

***Condition S* (Strong convexity condition)**

$$\sum_{|\alpha|=|\beta|=m/2} a_{\alpha+\beta} X_\alpha X_\beta > 0 \quad \text{in } \mathbb{R}^N \smallsetminus \{0\} \tag{1.5}$$

where N denotes the number of multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ of order $|\alpha| = m/2$ and $\{X_\alpha\}_{|\alpha|=m/2}$ stands for a generic point in $\mathbb{R}^N$.

Under these conditions the Green's function $G_\lambda(x)$ of $P_0(D) - \lambda$ verifies for $0 < \pm \arg \lambda < \pi$ an asymptotic formula of the form:

$$G_\lambda(x) = c_\pm \lambda^{\frac{n+1}{2m}-1} a(x) e^{\pm i\lambda^{1/m} Q_0(x)} \left(1 + O\left(1/|x|\right)\right) \tag{1.6}$$

as $|x| \to \infty$, uniformly in λ in any compact. Here $c_\pm$ are constants ($c_+ = \overline{c}_-$), $a(x)$ is a positive smooth homogeneous function of degree $-(n-1)/2$, and $Q_0(x)$ is a positive convex homogeneous function of degree 1 given by

$$Q_0(x) = \sup_{P_0(\xi)=1} \langle x, \xi \rangle,$$

(a more explicit expression of (1.6) is given in §4, formula (4.2)).

Note that in view of the homogeneity of $P_0(\xi)$ (1.6) can also be viewed as an asymptotic formula in λ (as λ tends suitably to infinity for a fixed $x \neq 0$).

Condition S is a strong convexity restriction. It was shown in [**1**] that Condition S implies in particular that the polynomial $P_0(\xi)$ is strictly convex, i.e.:

$$\operatorname{Hess} P_0(\xi) > 0 \quad \text{for } \xi \in \mathbb{R}^n \smallsetminus \{0\}. \tag{1.7}$$

In this connection note that under the assumption that the weaker condition (1.7) holds it can be shown that the asymptotic formula (1.6) is valid for the Green's functions $G_{\lambda \pm i0}(x)$ for $\lambda \in \mathbb{R}_+$. This follows from the explicit form of formula (1.4).

The asymptotic formula (1.6) is deduced in [**1**] from an asymptotic formula for the Green's function $G(x,t)$ of the parabolic operator $\partial/\partial t + P_0(D)$ as $t \to +0$. It was conjectured in [**1**] that this last asymptotic formula and consequently that the asymptotic formula (1.6) for $G_\lambda(x)$ should hold when Condition S is replaced by the weaker condition (1.7). In a later publication [**2**] it was shown by the authors that this conjecture is false for the Green's function of the parabolic operator.

In this paper we shall consider the following general problem. Find sufficient and necessary conditions in order that the Green's function $G(x)$ of a given elliptic operator $P(D)$, with $P(\xi) \neq 0$ on $\mathbb{R}^n$, possesses an asymptotic formula of the form:

$$G(x) = a(x) e^{iA(x)} (1 + o(1)) \tag{1.8}$$

as $|x| \to \infty$ in some infinite open cone Γ, where $A(x)$ is a smooth homogeneous function of degree 1 and $a(x)$ is a smooth homogeneous function of degree $-(n-1)/2$ in Γ.

The plan of this paper is as follows. In section 2 we describe some notions and preliminary results needed in the sequel. Our main theorem giving necessary and sufficient conditions for (1.8) to hold is discussed in section 2. In section 3 we describe applications of the main theorem to Green's functions of the operator $P_0(D) - \lambda$ where $P_0(D)$ is the operator in Theorem 1.1 with Condition S replaced by the condition that $P_0(\xi)$ is strictly convex. The main applications consist in giving necessary and sufficient conditions on the complex zeros of $P_0(\zeta) - \lambda$ in order that the Green's function $G_\lambda(x)$ will possess a nice asymptotic expansion.

In conclusion we observe that this paper is a revised version of a lecture given at the Journées Jean Leray on the occasion of the inauguration of the Laboratoire de Mathématiques Jean Leray at the University of Nantes. This is an expository paper with indications of proofs of the main results.

2. Preliminaries

In the following $P(D)$ denotes an elliptic operator with complex constant coefficients, of even order m, such that $P(\xi) \neq 0$ for $\xi \in \mathbb{R}^n$. $G(x)$ denotes the Green's function defined by (1.1).

With the polynomial $P(\zeta)$, $\zeta \in \mathbb{C}^n$, associate *norm functions* $K_P^*(x)$ and $K_P(x)$ on $\mathbb{R}^n$ defined as follows. For any unit vector $\theta \in \mathbb{R}^n$ set:

$$r(\theta) = \min\{t \in \mathbb{R}_+ : P(\xi + it\theta) = 0 \quad \text{for some } \xi \in \mathbb{R}^n\}.$$

Define

$$K_P^*(x) = \frac{|x|}{r\,(x/|x|)} \quad \text{for } x \in \mathbb{R}^n \smallsetminus \{0\}, \tag{2.1}$$

$K_P^*(0) = 0$, and set:

$$\Omega^* = \{x \in \mathbb{R}^n : K_P^*(x) < 1\}. \tag{2.1'}$$

Ω^* is a bounded open connected set in $\mathbb{R}^n$ containing the origin. Furthermore, since Ω^* is a connected component of the set: $\{\eta \in \mathbb{R}^n : P(\xi + i\eta) \neq 0, \forall\, \xi \in \mathbb{R}^n\}$ it follows by a known theorem that Ω^* is convex (see [**3**, p. 43]). Thus $K_P^*(x)$ is a convex homogeneous function of degree 1, $K_P^*(x) > 0$ for $x \neq 0$. Next define:

$$K_P(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi\rangle}{K_P^*(\xi)} = \sup_{\xi \in \partial\Omega^*} \langle x, \xi\rangle. \tag{2.2}$$

It is well known that $K_P(x)$, referred to as the *polar* of $K_P^*(x)$, is a positive convex homogeneous function of degree 1. Set:

$$\Omega = \{x \in \mathbb{R}^n : K_P(x) < 1\}.$$

Clearly, Ω is a convex open set containing the origin. The convexity of $K_P^*(x)$ implies that $K_P^*(x)$ is also the polar of $K_P(x)$, i.e.:

$$K_P^*(x) = \sup_{\xi \in \partial\Omega} \langle x, \xi\rangle. \tag{2.2'}$$

Next, observe that the Green's function of $P(D)$ verifies the following estimate:

$$|G(x)| \leqslant C|x|^m e^{-K_P(x)} \quad \text{for } |x| \geqslant 1, \tag{2.3}$$

C some constant.

We indicate the proof of the essentially known estimate (2.3). Pick a function $\chi(t) \in C^\infty(\mathbb{R})$ such that $\chi \equiv 0$ for $t \leqslant 1/2$, $\chi \equiv 1$ for $t \geqslant 1$. Set: $G_1(x) = \chi(|x|)G(x)$. Then $P(D)G_1 = f$ where $f \in C_0^\infty(\mathbb{R}^n)$. By Fourier transform:

$$G(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\widehat{f}(\xi)}{P(\xi)} e^{i\xi\cdot x} d\xi \quad \text{for } |x| \geqslant 1. \tag{2.4}$$

Noting that $\widehat{f}(\zeta)$ is an entire function in $\zeta \in \mathbb{C}^n$ which decays rapidly as $|\zeta| \to \infty$ in any tube: $|\operatorname{Im}\zeta| \leqslant R$, it follows by complex integration that in the integral (2.4) the domain of integration $\mathbb{R}^n$ can be shifted to the domain $\mathbb{R}^n + i(1 - 1/|x|)\omega^*$ where ω^* is any point in $\partial\Omega^*$. An easy estimation of the resulting integral yields:

$$|G(x)| \leqslant C|x|^m e^{-\langle\omega^*, x\rangle} \quad \text{for } |x| \geqslant 1, \tag{2.5}$$

C some constant independent of x or ω^*. Minimizing the r.h.s. of (2.5) with respect to ω^* yields (2.3).

The following (essentially well known) proposition shows that the estimate (2.5) is quite precise in the exponential factor.

Proposition 2.1. — *Suppose that $G(x)$ verifies an estimate of the form:*

$$|G(x)| \leqslant C|x|^N e^{-Q(x)} \quad \textit{for } |x| \geqslant 1,$$

where $Q(x)$ is some continuous homogeneous function of degree 1 on $\mathbb{R}^n \smallsetminus \{0\}$. Then

$$Q(\omega) \leqslant K_P(\omega)$$

at all points $\omega \in \partial\Omega$ which are extremal points of $\overline{\Omega}$.

We conclude this section with some notions and definitions related to the boundaries of the conjugate convex sets Ω and Ω^*.

Let Γ be an infinite open convex cone in $\mathbb{R}^n$ with vertex at the origin. Consider the boundary set:

$$\partial\Omega_\Gamma := \partial\Omega \cap \Gamma. \tag{2.6}$$

Assume that $\partial\Omega_\Gamma$ is a C^2 manifold with a positive Gaussian curvature at every point (so that $K_P(x)$ is a C^2 function and $\operatorname{Hess} K_P(x)^2 > 0$ in Γ). Define:

$$\Gamma^* = \{x \in \mathbb{R}^n \smallsetminus \{0\} : x/|x| = K_P'(y)/|K_P'(y)| \quad \text{for some } y \in \Gamma\},$$

(here $K_P'(y) := \nabla K_P(y)$). Γ^* is an open convex cone which we shall refer to as the polar to Γ with respect to the "norm" $K_P(x)$. One finds readily that for $x \in \Gamma^*$:

$$K_P^*(x) = \langle x, \omega(x)\rangle \tag{2.7}$$

where $\omega(x)$ is the unique point in $\partial\Omega_\Gamma$ such that $K_P'(\omega(x))$ is in the direction of x. From (2.7) it follows that $K_P^*(x)$ is a C^2 function in Γ^* and setting:

$$\partial\Omega^*_{\Gamma^*} = \partial\Omega^* \cap \Gamma^* \tag{2.6*}$$