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ON THE CLASSIFICATION OF CUBIC PLANAR CREMONA MAPS

BY ALBERTO CALABRI & NGUYEN THI NGOC GIAO

ABSTRACT. — We give a fine and complete classification of cubic planar Cremona maps, up to automorphisms of the plane. For this purpose, we introduce a new discrete invariant for cubic planar Cremona maps, called enriched weighted proximity graph, which encodes some properties of the base locus of the Cremona map.

RÉSUMÉ (*Sur la classification des transformations cubiques planes de Cremona*). — Nous donnons une classification complète et fine des transformations cubiques planes de Cremona, modulo l'action des automorphismes du plan. Dans ce but, nous introduisons un nouvel invariant discret pour les transformations cubiques planes de Cremona, appelé graphe de proximité pondéré enrichi, qui décrit certaines propriétés du lieu de base de la transformation de Cremona.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. We denote by $\mathbb{P}^2$ the projective plane and by $\text{Bir}(\mathbb{P}^2)$ the *plane Cremona group*, that is, the group of birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Generators of $\text{Bir}(\mathbb{P}^2)$ are very well known, for over a century, thanks to the classical works of Noether and Castelnuovo, namely

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$\text{Bir}(\mathbb{P}^2)$ is generated by automorphisms of $\mathbb{P}^2$ and a single quadratic transformation, e.g., the so-called elementary quadratic map

$$(1) \quad \sigma: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x : y : z] \mapsto [yz : xz : xy].$$

Nonetheless, the plane Cremona group is currently still a very active research area. Very recent remarkable results in this subject are, e.g., a new presentation of $\text{Bir}(\mathbb{P}^2)$, due to Urech and Zimmermann in [14], and the study by Blanc and Furter in [3] of the so-called *length* of a plane Cremona map φ , that is, the minimum number of *Jonquières* maps needed to decompose φ ; see Definition 5.3 for Jonquières maps.

Since all cubic planar Cremona maps trivially have length 1, let us introduce two refinements of the notion of length, namely the *quadratic length* (or *ordinary quadratic length*) of a plane Cremona map φ , that is, the minimum number of quadratic (or ordinary quadratic) maps needed to decompose φ , where an ordinary quadratic map is such that its three base points are all proper, cf. Section 5 for more details.

Let us say that two plane Cremona maps $\varphi, \varphi': \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are *equivalent* if there exist two automorphisms $\alpha, \alpha' \in \text{Aut}(\mathbb{P}^2)$, such that $\varphi' = \alpha' \circ \varphi \circ \alpha$. It is very well known from the beginning of the study of plane Cremona maps, more than one hundred years ago, that there are exactly three equivalence classes of quadratic planar Cremona maps, which can be classified by the number of proper base points of the map; cf. Remark 5.2.

A classification of equivalence classes of cubic planar Cremona maps was described only few years ago by Cerveau and Déserti in [5]; they found 32 types of cubic planar Cremona maps, namely 27 types that are a single map, four types that are families of maps depending on one parameter, and one type that is a family of maps depending on two parameters. Their classification is based on the detailed analysis of those plane curves that are contracted by a cubic planar Cremona map.

However, it turns out that the classification in [5] is not complete, and it contains some inaccuracies; see Section 8 for a more detailed account.

- We found a map (our type 15 in Table 1.2) that does not occur in their list.
- We found that their type 17, that is, a single map, should be replaced by a one-parameter set of maps (our type 28 in Table 1.2).
- We found that their type 19 is equivalent to a particular case of their type 18.
- We found that their type 31 is equivalent to a particular case of their type 30.

Furthermore, in [5] it is not clear when two maps of the same type, that depend on parameters, are equivalent.

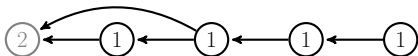
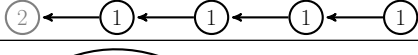
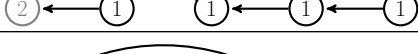
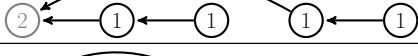
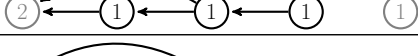
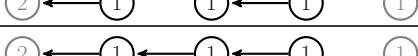
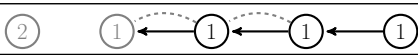
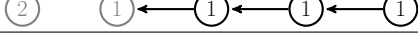
Our main result in this paper is a fine classification of equivalence classes of cubic planar Cremona maps. For this purpose, we introduce a new discrete invariant, which we call *enriched weighted proximity graph* of a cubic planar Cremona map φ , determined by the properties of the base points of the homaloidal net $\mathcal{L}_\varphi$ defining φ . Namely, it is a directed graph with five vertices: one vertex p_1 with weight 2 corresponding to the base point of multiplicity 2 of $\mathcal{L}_\varphi$ and the other four vertices $p_2, \dots, p_5$ with weight 1 corresponding to the other four (proper or infinitely near) base points of multiplicity 1 of $\mathcal{L}_\varphi$, such that

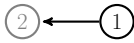
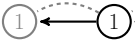
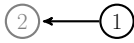
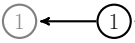
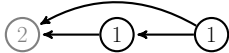
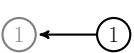
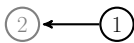
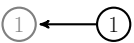
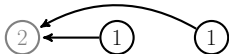
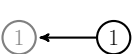
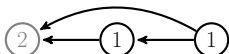
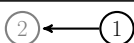
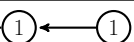
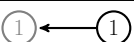
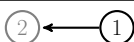
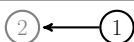
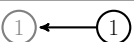
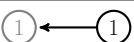
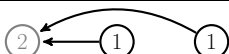
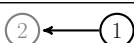
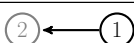
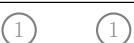
- there is an arrow starting from a vertex p_i towards a vertex p_j if and only if p_i is *proximate* to p_j ; cf. Definition 2.6;
- there is a dashed line joining three vertices if and only if the corresponding three base points are aligned.

In other words, the enriched weighted proximity graph takes into account the proximity relations among the base points, their multiplicities and their relative position; cf. Section 6. Our first result is their classification.

THEOREM 1.1. — *There are exactly 31 enriched weighted proximity graphs of cubic planar Cremona maps, up to isomorphism, listed in Table 1.1.*

Table 1.1: Enriched weighted proximity graphs and ordinary quadratic lengths of cubic planar Cremona maps.

#	Enriched weighted prox. graph	oql
1		6
2		5
3		5
4		4
5		5
6		4
7		4
8		5
9		4

10			3
11			3
12			3
13			3
14			3
15			3
16			3
17			4
18			3
19			3
20			3
21			2
22			3
23			2
24			3
25			2
26			2
27			2
28			3
29			2
30			2
31			2

Correspondingly, we find 31 types of equivalence classes of cubic planar Cremona maps. Before stating our classification theorem, we need a bit of notation.

Let us set $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$ and let us define the following maps:

$$g_1, g_2: \mathbb{C}^{**} \times \mathbb{C}^{**} \rightarrow \mathbb{C}^{**} \times \mathbb{C}^{**}, \quad g_1(a, b) = (b, a), \quad g_2(a, b) = \left(\frac{1}{a}, \frac{1}{b}\right).$$

Therefore, $g_3 := g_2 \circ g_1 = g_1 \circ g_2$ is the map $(a, b) \mapsto (1/b, 1/a)$. Clearly,

$$G = \{\text{id}, g_1, g_2, g_3\}$$

is a group, under the composition, which is isomorphic to $((\mathbb{Z}/2\mathbb{Z})^2, +)$.

For $a \neq b$ and $a, b \in \mathbb{C}^{**}$, let us denote by S' the following set

$$S' = \left\{ (a, b), \left(\frac{a}{a-1}, \frac{a-b}{a-1}\right), \left(\frac{b}{b-1}, \frac{b-a}{b-1}\right), \right. \\ \left. \left(\frac{a-b}{b(a-1)}, \frac{1}{1-a}\right), \left(\frac{b-a}{a(b-1)}, \frac{1}{1-b}\right), \left(\frac{a-1}{b-1}, \frac{b(a-1)}{a(b-1)}\right) \right\}$$

and let us define

$$(2) \quad S = \{g(s) \mid g \in G \text{ and } s \in S'\}.$$

THEOREM 1.2. — *Any cubic planar Cremona map is equivalent to one of the maps in Table 1.2, where the first 25 types are single maps, types 26–30 depend on one parameter $\gamma \neq 0, 1$ and type 31 depends on two parameters a, b , where $a, b \neq 0, 1$ and $a \neq b$.*

Two cubic planar Cremona maps of two different types are not equivalent.

Concerning the types depending on parameters:

- $\varphi_{26, \gamma}$, that is type 26 in Table 1.2 with parameter $\gamma \neq 0, 1$, is equivalent to $\varphi_{26, \gamma'}$ if and only if either $\gamma' = \gamma$ or $\gamma' = \gamma/(\gamma - 1)$;
- $\varphi_{27, \gamma}$, that is type 27 in Table 1.2 with parameter $\gamma \neq 0, 1$, is equivalent to $\varphi_{27, \gamma'}$ if and only if either $\gamma' = \gamma$ or $\gamma' = 1/\gamma$;
- for $n \in \{28, 29, 30\}$, the map $\varphi_{n, \gamma}$, that is type n in Table 1.2 with parameter $\gamma \neq 0, 1$, is equivalent to $\varphi_{n, \gamma'}$ if and only if

$$\gamma' \in \left\{ \gamma, \frac{1}{\gamma}, 1 - \gamma, \frac{1}{1 - \gamma}, \frac{\gamma}{\gamma - 1}, \frac{\gamma - 1}{\gamma} \right\};$$

- $\varphi_{31, a, b}$, that is type 31 in Table 1.2 with two parameters $a, b \neq 0, 1$, $a \neq b$, is equivalent to $\varphi_{31, a', b'}$ if and only if $(a', b') \in S$, where S is defined in (2).

In Table 1.2, the first column lists our type, the second column lists the formula of the map, the third column lists the corresponding type in [5], cf. Section 8, and finally the fourth column lists the type of the inverse map.

As a first application of our classification, we compute the ordinary quadratic length and the quadratic length of all cubic planar Cremona maps.

TABLE 1.2. Types of cubic planar Cremona maps.

#	Map	[5]	Inv.
1	$[xz^2 + y^3 : yz^2 : z^3]$	1	1
2	$[x(x^2 + yz) : y^3 : y(x^2 + yz)]$	20	8
3	$[xz^2 : x^3 + xyz : z^3]$	3	5
4	$[x^2z : x^3 + z^3 + xyz : xz^2]$	4	4
5	$[x^2z : x^2y + z^3 : xz^2]$	5	3
6	$[x^2(x - y) : xy(x - y) : xyz + y^3]$	12	6
7	$[x(x^2 + yz) : y(x^2 + yz) : xy^2]$	24	17
8	$[xyz : yz^2 : z^3 - x^2y]$	6	2
9	$[y^2z : x(xz + y^2) : y(xz + y^2)]$	21	9
10	$[x^3 : y^2z : xyz]$	7	10
11	$[x(y^2 + xz) : y(y^2 + xz) : xyz]$	22	18
12	$[xz^2 : x^2y : z^3]$	2	12
13	$[x(y^2 + xz) : y(y^2 + xz) : xy^2]$	23	20
14	$[x^3 : x^2y : (x - y)yz]$	11	15
15	$[x^2y : xy^2 : (x - y)^2z]$	($\star$)	14
16	$[x(x^2 + yz) : y(x^2 + yz) : xy(x - y)]$	28	24
17	$[xyz : y^2z : x(y^2 - xz)]$	10	7
18	$[x^2(y - z) : xy(y - z) : y^2z]$	8	11
19	$[x(x^2 + yz + xz) : y(x^2 + yz + xz) : xyz]$	26	19
20	$[x^2z : xyz : y^2(x - z)]$	9	13
21	$[x(xy + xz + yz) : y(xy + xz + yz) : xyz]$	25	21
22	$[xz(x + y) : yz(x + y) : xy^2]$	13	22
23	$[x(x^2 + xy + yz) : y(x^2 + xy + yz) : xyz]$	27	25
24	$[xyz : (y - x)yz : x(x - y)(y - z)]$	15	16
25	$[x(x + y)(y + z) : y(x + y)(y + z) : xyz]$	14	23
26	$[x(\gamma xz - \gamma y^2 - xy + y^2) : \gamma xy(z - y) : \gamma y^2(z - x)]$	29	26
27	$[\gamma x^2y : \gamma xy^2 : (x + y)(x + \gamma y)z]$	16	27
28	$[xy(x - y) : xz(y - \gamma x) : z(y + \gamma x)(y - \gamma x)]$	$17^\dagger$	28
29	$[xy(x - y) : x(xy - \gamma xy + \gamma xz - yz) : x^2y - \gamma^2x^2y + \gamma^2x^2z - y^2z]$	30	30
30	$[x(xy + \gamma xz - xz - \gamma y^2) : \gamma xz(x - y) : \gamma z(x - y)(x + y)]$	18	29
31	$[ax(-abxz + aby^2 - b^2xy + b^2xz + axy - ay^2)$ $: ax(-abxz + abyz + axy - ayz - bxy + bxz)$ $: -a^2bx^2z + a^2by^2z + a^2x^2y - a^2y^2z - b^2x^2y + b^2x^2z]$	32	31