

quatrième série - tome 51 fascicule 4 juillet-août 2018

*ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE*

Javier CARVAJAL-ROJAS & Karl SCHWEDE & Kevin TUCKER

Fundamental groups of F -regular singularities via F -signature

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Responsable du comité de rédaction / *Editor-in-chief*

Patrick BERNARD

Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE
de 1883 à 1888 par H. DEBRAY
de 1889 à 1900 par C. HERMITE
de 1901 à 1917 par G. DARBOUX
de 1918 à 1941 par É. PICARD
de 1942 à 1967 par P. MONTEL

Comité de rédaction au 1^{er} mars 2018

P. BERNARD	A. NEVES
S. BOUCKSOM	J. SZEFTTEL
R. CERF	S. VŨ NGỌC
G. CHENEVIER	A. WIENHARD
Y. DE CORNULIER	G. WILLIAMSON
E. KOWALSKI	

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 540 €. Hors Europe : 595 € (\$ 863). Vente au numéro : 77 €.

© 2018 Société Mathématique de France, Paris

En application de la loi du 1^{er} juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).

All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 n^{os} / an

FUNDAMENTAL GROUPS OF F -REGULAR SINGULARITIES VIA F -SIGNATURE

BY JAVIER CARVAJAL-ROJAS, KARL SCHWEDE
AND KEVIN TUCKER

ABSTRACT. – We prove that the local étale fundamental group of a strongly F -regular singularity is finite. These results are analogous to results of Xu and Greb-Kebekus-Peternell for KLT singularities in characteristic 0. Our result is effective, we show that the reciprocal of the F -signature of the singularity gives a bound on the size of this fundamental group. To prove these results we develop new transformation rules for the F -signature under finite étale-in-codimension-one extensions. We also obtain purity of the branch locus over rings with mild singularities (particularly if the F -signature is $> 1/2$).

RÉSUMÉ. – Nous montrons que le groupe fondamental local étale d’une singularité F -régulière est fini. Ce théorème représente l’analogie en caractéristique p des résultats obtenus par Xu et Greb-Kebekus-Peternell pour les singularités KLT. Nous montrons que le cardinal du groupe fondamental est majoré par l’inverse de la F -signature de la singularité. En particulier, notre résultat principal est effectif. Pour cela, nous établissons des nouvelles formules de transformation de la F -signature par rapport aux extensions étale en codimension un. Nous obtenons également un nouveau critère de pureté du lieu de branchement sur les anneaux à singularités faibles. Ceci s’applique en particulier aux anneaux dont la F -signature est supérieure à $1/2$.

1. Introduction

In [21, Question 26] J. Kollár asked whether if $(0 \in X)$ is the germ of a KLT singularity, then $\pi_1(X \setminus \{0\})$ is finite. In [41] C. Xu showed that this holds for the étale local fundamental group, in other words, for the profinite completion of $\pi_1(X \setminus \{0\})$. Building on this result, [11] proved the finiteness of the étale fundamental groups of the regular locus of KLT singularities (see also [35]). Over the past few decades, we have learned that KLT singularities are closely related to strongly F -regular singularities in characteristic $p > 0$, see [16, 15]. Hence it is natural to ask whether their local étale fundamental groups are also finite. We show that

The first named author was supported in part by the NSF FRG Grant DMS #1265261/1501115. The second named author was supported in part by the NSF FRG Grant DMS #1265261/1501115 and NSF CAREER Grant DMS #1252860/1501102. The third named author was supported in part by NSF Grants DMS #1419448, DMS #1602070, and a fellowship from the Sloan foundation.

this is indeed the case. In fact, we find an upper bound for the size of the fundamental group in terms of a well studied invariant for measuring singularities in characteristic $p > 0$, the F -signature $s(R)$.

THEOREM A (Theorem 5.1). – *Let $(R, \mathfrak{m}, k)$ be a normal F -finite and strongly F -regular strictly Henselian⁽¹⁾ local domain of prime characteristic $p > 0$, with dimension $d \geq 2$. Then the étale fundamental group of the punctured spectrum of R , i.e., $\pi_1 := \pi_1^{\text{ét}}(\text{Spec}^\circ(R))$, is finite. Furthermore, the order of π_1 is at most $1/s(R)$ and is prime to p . The same also holds for $\pi_1^{\text{ét}}(\text{Spec}(R) \setminus Z)$ where $Z \subseteq \text{Spec } R$ has codimension ≥ 2 .*

Observe that unlike the characteristic zero situation, our characteristic $p > 0$ result is effective. We give an explicit bound on the size of the π_1 . It is also worth noting that we are working with the étale fundamental group, not the *tame* fundamental group. Indeed, for R strongly F -regular, any finite Galois étale in codimension 1 local extension $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, \ell)$ must be tame everywhere. This was already implicitly observed in [31] but we make it precise here. Indeed, we note that p cannot divide $[K(S) : K(R)]$ if the residue fields are equal (Corollary 2.11).

The technical tool where F -regularity is used in our proof is a transformation rule for F -signature under finite étale-in-codimension-1-morphisms. The F -signature was introduced implicitly in [32] and explicitly in [18]. Roughly speaking, it measures how many different ways $R \hookrightarrow F_*^e R$ splits as e goes to infinity. Explicitly, if R has perfect residue field and $F_*^e R = R^{\oplus a_e} \oplus M$ as an R -module, where M has no free R -summands, then $s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{e \dim R}}$. Here are three quick facts:

- The limit $s(R)$ exists [37].
- $s(R) > 0$ if and only if R is strongly F -regular [1].
- $s(R) \leq 1$.

Note that there have been a number of transformation rules for F -signature under finite maps in the past. However, they were generally only an inequality (that went the wrong way for our purposes), or assumed that S is flat over R (or made other assumptions about R and S). See for instance [18, 42, 17, 37].

THEOREM B (Theorem 3.1). – *Let $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, \ell)$ be a module-finite local extension of F -finite d -dimensional normal local domains in characteristic $p > 0$, with corresponding extension of fraction fields $K \subseteq L$. Suppose $R \subseteq S$ is étale in codimension 1, and that R is strongly F -regular. Then if one writes $S = R^{\oplus f} \oplus M$ as a decomposition of R -modules so that M has no nonzero free direct summands, then $f = [\ell : k] \geq 1$ and the following equality holds:*

$$s(S) = \frac{[L : K]}{[\ell : k]} \cdot s(R).$$

Below, before Theorem D, we discuss how to still get precise transformation rules of F -signature even when $R \subseteq S$ is not necessarily étale in codimension 1.

By applying Theorem B in the case $k = \ell$, we see that $s(S) = [L : K] \cdot s(R)$. Since $s(S) \leq 1$, we immediately see that $[L : K] \leq 1/s(R)$. In other words, the reciprocal

⁽¹⁾ This just means it is Henselian with separably closed residue field.

of the F -signature $s(R)$ gives an upper bound on the generic rank of a finite local étale in codimension 1 extension with the same residue field. Theorem A then follows. We also obtain characteristic $p > 0$ corollaries similar to some of those in [11].

Because our bound on the size of the étale fundamental group is effective, we immediately obtain a new result on purity of the branch locus.

THEOREM C (Corollary 3.3). – *Suppose $Y \rightarrow X$ is a finite dominant map of F -finite normal integral schemes. If $s(\mathcal{O}_{X,P}) > 1/2$ for all $P \in X$ then the branch locus of $Y \rightarrow X$ has no irreducible components of codimension ≥ 2 , in other words it is a divisor.*

In [5], the notion of F -signature of pairs was introduced. In Theorem 4.4, we obtain an analogous result to Theorem B in the context of pairs. Indeed, if (R, Δ) is a strongly F -regular pair, then this can be interpreted as follows. The reciprocal of $s(R, \Delta)$ gives an upper bound on the generic rank of a finite local extension $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ such that $\pi^* \Delta - \text{Ram} \geq 0$ (here Ram is the ramification divisor on $\text{Spec } S$ and $\pi : \text{Spec } S \rightarrow \text{Spec } R$ is the induced map). By taking cones, this immediately yields the following characteristic $p > 0$ analog of the second main result of [41]. Here note that globally F -regular varieties are an analog of log-Fano varieties in characteristic zero [30].

THEOREM D (Corollary 4.8). – *Suppose that (X, Δ) is a globally F -regular projective pair over an algebraically closed field of characteristic $p > 0$. There is a number n such that every finite separable cover $\pi : Y \rightarrow X$ with $\pi^* \Delta - \text{Ram} \geq 0$ has generic rank $[K(Y) : K(X)] \leq n$.*

Acknowledgements: The authors would like to thank János Kollár, Christian Liedtke, Linquan Ma, Lance Miller, Mircea Mustață, Stefan Patrikis and David Speyer for valuable and inspiring conversations. We would like to thank Stefan Kebekus, Lance Miller, Mihai Păun and Chenyang Xu for valuable comments on previous drafts. We would especially like to thank David Speyer for sharing an early preprint of [33] with us and allowing us to include Lemma 2.14 which is a special case of his result.

2. Preliminaries

CONVENTION 2.1. – Throughout this paper, all rings will be assumed to be Noetherian. They will all be characteristic $p > 0$ unless otherwise stated and they will all be F -finite. All schemes will be assumed to be Noetherian and separated. If R is an integral domain, then $K(R)$ will denote the fraction field of R (likewise with $K(X)$ if X is an integral scheme). Given a finite separable map of normal integral schemes $f : Y \rightarrow X$, we use Ram to denote the ramification divisor on Y .