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CORRECTIONS TO *UPPER BOUNDS FOR COURANT-SHARP NEUMANN AND ROBIN EIGENVALUES*

BY KATIE GITTINS & CORENTIN LÉNA

ABSTRACT. — In [2], the number of nodal domains for the eigenfunctions of the Neumann or Robin Laplacian is bounded from above using the classical (Euclidean) Faber–Krahn inequality. However, in an arbitrary Riemannian manifold, this inequality might not hold. We supply the missing arguments in two dimensions and outline a modification of the method, which preserves most of the results, in n dimensions.

RÉSUMÉ (*Corrections à « Majoration des valeurs propres Courant strictes de Neumann et Robin »*). — Dans [2], on obtient une majoration du nombre de domaines nodaux pour une fonction propre du laplacien de Neumann ou de Robin à l'aide de l'inégalité de Faber-Krahn classique (euclidienne). Cette inégalité n'est toutefois pas nécessairement vérifiée dans une variété riemannienne quelconque. La présente note donne les arguments manquants en deux dimensions et indique à grands traits une modification de la méthode en dimension n qui préserve l'essentiel des résultats.

The mapping F introduced in Equation (6) on page 105 should instead be defined from the cylinder $\mathcal{C}_L := (\mathbb{R}/(L\mathbb{Z}))\times\mathbb{R}$ to $\mathbb{R}^2$. The set V^R, defined at the bottom of page 109 through a preimage by F, is, therefore, an open set in $\mathcal{C}_L$

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and not in $\mathbb{R}^2$. It is not a priori clear that the Faber–Krahn inequality with the same constant as in the Euclidean case (hereafter called *classical Faber–Krahn inequality*) holds for V^R. The same gap occurs in Section 5.3 when considering a possibly multiply connected domain in $\mathbb{R}^2$, for which V^R is an open set in the disjoint union $\bigcup_{i=0}^{b} \mathcal{C}_{L_i}$. Similarly, when $\Omega \subset \mathbb{R}^n$, we implicitly apply (on page 123) the classical Faber–Krahn inequality in the cylindrical manifold $N(\Gamma) = \Gamma \times \mathbb{R}$, where it might not hold.

When Ω is a (possibly multiply connected) C^2 domain in $\mathbb{R}^2$, we can fix the issue by combining a few known geometric results. Examination of the proof of the classical Faber–Krahn inequality by symmetrization (see, for instance, [1, I.9]) shows that it holds provided that all open sets $W \subset V^R$ satisfy the classical isoperimetric inequality $|\partial W|^2 \geq 4\pi|W|$. From [3, §6], the classical isoperimetric inequality holds for any open set $W_i \subset \mathcal{C}_{L_i}$ such that $|W_i| \leq L_i^2/\pi$. Note that the parameter a in [3] is the radius of the cylinder, equal to $L_i/2\pi$ in our notation. We then remark that V^R can be written as a disjoint union $\bigcup_{i=0}^{b} V_i$, with each V_i contained in

$$(\mathbb{R}/(L_i\mathbb{Z})) \times (-\delta_0(\Omega), \delta_0(\Omega)) \subset \mathcal{C}_{L_i}$$

(some V_i's may be empty). In addition, $\delta_0(\Omega)$ is less than or equal to the smallest radius of curvature of $\partial\Omega$, denoted by $t_+(\Omega)$. We can write $\Omega = \Omega_0 \setminus \bigcup_{i=1}^{b} \overline{D_i}$, where Ω_0 and all the D_i's are simply connected C^2 domains and apply to each the inequality $\pi t_+(D)^2 \leq |D|$, valid for any simply connected domain $D \subset \mathbb{R}^2$ (see [4] and references therein). We obtain

$$\delta_0(\Omega) \leq t_+(\Omega) \leq \frac{1}{\pi^{1/2}} \min\left\{|\Omega_0|^{1/2}, \min_{1\leq i\leq b} |D_i|^{1/2}\right\} \leq \frac{1}{2\pi} \min_{0\leq i\leq b} L_i$$

(where the last inequality follows from the isoperimetric inequality in $\mathbb{R}^2$). Thus, $|V_i| \leq 2\,\delta_0(\Omega)\,L_i \leq L_i^2/\pi$ for all i, and any $W_i \subset V_i$ satisfies the isoperimetric inequality. This is, therefore, also the case for any $W = \bigcup_{i=0}^{b} W_i \subset V^R$, which concludes the proof. All the results of Sections 2 to 8 hold without any change.

Most of the material in Section 9 can be recovered by replacing the quantities $\delta_+(x')$, $\underline{\delta}_+(\Omega)$ and $\delta_0(\Omega)$, defined on page 121, with, respectively,

$$\begin{aligned}
\delta(x') &:= \sup\{\delta > 0 : \operatorname{dist}(F(x',t), \partial\Omega) = |t| \text{ for all } t \in [-\delta, \delta]\},\\
\underline{\delta}(\Omega) &:= \inf_{x'\in\partial\Omega} \delta(x'),\\
\delta_1(\Omega) &:= \min\{t_+(\Omega), \underline{\delta}(\Omega)\}.
\end{aligned}$$

Having defined the set $V^R \subset \Gamma \times \mathbb{R}$ as in Section 5.2, we modify our method by taking the additional steps of considering the image $U^R := F(V^R) \subset \mathbb{R}^n$ and applying the Faber–Krahn inequality to U^R. We get an upper bound on the number of boundary nodal domains $\nu_1(\varepsilon_0, u)$, and thus find results similar to Proposition 9.2, except that $\delta_0(\Omega)$ is replaced with $\delta_1(\Omega)$, and the constants depending only on n are larger ones. Note that the geometric quantity $\delta_1(\Omega)$ depends on the cut-distance to the boundary (also known as the injectivity radius of the normal exponential map) with respect to both the interior and the exterior of the domain Ω, whereas $\delta_0(\Omega)$ depends only on the interior. If, in addition, Ω is assumed to be convex, then $\delta_1(\Omega) = \delta_0(\Omega)$, since $\mathrm{dist}(F(x', t), \partial\Omega) = -t$ for all $x' \in \partial\Omega$ and $t \leq 0$ (the cut-distance with respect to the exterior is infinite). Therefore, in that case, only the constants depending on n change. In particular, Proposition 9.4 holds as written.

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