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PSEUDO-SPLIT FIBERS AND ARITHMETIC SURJECTIVITY

BY DANIEL LOUGHRAN, ALEXEI N. SKOROBOGATOV
AND ARNE SMEETS

ABSTRACT. — Let $f : X \rightarrow Y$ be a dominant morphism of smooth, proper and geometrically integral varieties over a number field k , with geometrically integral generic fiber. We give a necessary and sufficient geometric criterion for the induced map $X(k_v) \rightarrow Y(k_v)$ to be surjective for almost all places v of k . This generalizes a result of Denef which had previously been conjectured by Colliot-Thélène, and can be seen as an optimal geometric version of the celebrated Ax-Kochen theorem.

RÉSUMÉ. — Soit $f : X \rightarrow Y$ un morphisme dominant de variétés lisses, propres et géométriquement intègres définies sur un corps de nombres k , dont la fibre générique est géométriquement intègre. Nous donnons un critère géométrique, à la fois nécessaire et suffisant, pour que l'application induite $X(k_v) \rightarrow Y(k_v)$ soit surjective pour presque toute place v de k . Ceci généralise un résultat de Denef précédemment conjecturé par Colliot-Thélène. Notre résultat peut être vu comme une version géométrique optimale du célèbre théorème de Ax-Kochen.

1. Introduction

1.1. — A famous theorem of Ax-Kochen [6] states that any homogeneous polynomial over $\mathbf{Q}_p$ of degree d in at least $d^2 + 1$ variables has a non-trivial zero, provided that p avoids a certain finite exceptional set of primes depending only on d . This was originally proved using model theory. Denef recently found purely algebro-geometric proofs [12, 13]. In [13], he did so by proving a more general conjecture of Colliot-Thélène [8, §3, Conjecture].

The essential notion (first introduced by the second author in [32, Definition 0.1]) appearing in this conjecture is that of a *split scheme*:

DEFINITION 1.1. — Let k be a perfect field. A scheme X of finite type over k is called *split* if X contains an irreducible component of multiplicity 1 which is geometrically irreducible.

Here the *multiplicity* of an irreducible component Z of X is the length of the local ring of X at the generic point of Z . In particular, it has multiplicity 1 if and only if it is generically reduced. Denef's result [13, Theorem 1.2] is the following.

THEOREM 1.2 (Denef). – *Let $f : X \rightarrow Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over a number field k , with geometrically integral generic fiber. Assume that for every modification $f' : X' \rightarrow Y'$ of f with X' and Y' smooth such that the generic fibers of f and f' are isomorphic, the fiber $(f')^{-1}(D)$ is a split $\kappa(D)$ -variety for every $D \in (Y')^{(1)}$.*

Then $Y(k_v) = f(X(k_v))$ for all but finitely many places v of k .

Here k_v denotes the completion of k at the place v , $(Y')^{(1)}$ denotes the set of points of codimension 1 in Y' , and $\kappa(D)$ is the residue field of D . A *modification* of f is a commutative diagram

$$(1.1) \quad \begin{array}{ccc} X' & \xrightarrow{\alpha_X} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\alpha_Y} & Y, \end{array}$$

where $f' : X' \rightarrow Y'$ is a dominant morphism of proper and geometrically integral varieties over k , and $\alpha_X : X' \rightarrow X$ and $\alpha_Y : Y' \rightarrow Y$ are birational morphisms.

One obtains the Ax-Kochen theorem by applying Theorem 1.2 to the universal family of all hypersurfaces of degree d in $\mathbf{P}^n$ with $n \geq d^2$; that the hypotheses of the theorem are satisfied in this case was shown by Colliot-Thélène (see [8, Remarque 4]).

1.2. – In this paper we strengthen Denef's result, by determining conditions which are both *necessary and sufficient* for the map $f : X(k_v) \rightarrow Y(k_v)$ to be surjective for almost all places v . Our result uses the following weakening of Definition 1.1 (in §2.2 we also give a more general definition over arbitrary ground fields).

DEFINITION 1.3. – Let k be a perfect field with algebraic closure $\bar{k}$. A scheme X of finite type over k is called *pseudo-split* if every element of $\text{Gal}(\bar{k}/k)$ fixes some irreducible component of $X \times_k \bar{k}$ of multiplicity 1.

It is clear that pseudo-splitness is weaker than splitness, the latter meaning that a *single* irreducible component of $X \times_k \bar{k}$ of multiplicity 1 is fixed by *all* of $\text{Gal}(\bar{k}/k)$. With this terminology, we can state our generalization of Denef's result as follows:

THEOREM 1.4. – *Let k be a number field. Let $f : X \rightarrow Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over k with geometrically integral generic fiber. Then $Y(k_v) = f(X(k_v))$ for all but finitely many places v of k if and only if for every modification $f' : X' \rightarrow Y'$ of f , with X' and Y' smooth, and for every point $D \in (Y')^{(1)}$, the fiber $(f')^{-1}(D)$ is a pseudo-split $\kappa(D)$ -variety.*

In the notation introduced by the first and third named authors in their recent work [25, §3], the morphisms $f : X \rightarrow Y$ satisfying the conclusion of the theorem are exactly the morphisms such that $\Delta(f') = 0$ for every modification f' of f .

1.3. – Theorem 1.4 will be deduced from finer results. With $f : X \rightarrow Y$ as in Theorem 1.2, Colliot-Thélène asked in [9, §13.1] how the geometry of f relates to the surjectivity of the map $X(k_v) \rightarrow Y(k_v)$, for a possibly infinite collection of places v . He called this phenomenon “surjectivité arithmétique” (note that this is different from the notion of arithmetic surjectivity studied in [16]). We develop general criteria which allow one to decide whether, for an *individual* (but large) place v , the map $X(k_v) \rightarrow Y(k_v)$ is surjective. They involve certain invariants which we call “ s -invariants,” defined in §3—local versions of the δ -invariants introduced in [25, §3]; their definition is given in terms of the geometry of f and does not involve model theory.

The following result is proved in §6 using tools from logarithmic geometry, in particular, a logarithmic version of Hensel’s lemma and “weak toroidalisation”. It should be viewed as the main theorem of the paper and is a geometric criterion, in the style of Colliot-Thélène’s conjecture, for surjectivity of the map $X(k_v) \rightarrow Y(k_v)$.

THEOREM 1.5. – *Let k be a number field. Let $f : X \rightarrow Y$ be a dominant morphism of smooth, proper, geometrically integral varieties over k , with geometrically integral generic fiber. Then there exist a modification $f' : X' \rightarrow Y'$ of f with X' and Y' smooth, and a finite set of places S of k such that for all $v \notin S$ the following are equivalent:*

- (1) *the map $X(k_v) \rightarrow Y(k_v)$ is surjective;*
- (2) *for every codimension 1 point $D' \in (Y')^{(1)}$, we have $s_{f', D'}(v) = 1$.*

The invariants $s_{f', D'}(v)$ appearing in the statement will be defined in §3. They are defined in terms of the Galois action on the irreducible components of the fiber of f' over D' . One benefit of our approach is that it yields a single model for f which can be used to test arithmetic surjectivity using a finite list of criteria.

A simple consequence of Theorem 1.5 is the following:

THEOREM 1.6. – *Let $f : X \rightarrow Y$ be a dominant morphism of smooth, proper and geometrically integral varieties over a number field k , with geometrically integral generic fiber. The set of places v such that $Y(k_v) = f(X(k_v))$ is Frobenian.*

Here we use the term “Frobenian” in the sense of Serre [31, §3.3] (see §3.1). Frobenian sets of places have a density, but being Frobenian is much stronger than just having a density; for example, an infinite Frobenian set has positive density. It is also possible to prove Theorem 1.6 using model-theoretic results and techniques such as quantifier elimination [5, 28]; our method avoids these and is completely algebro-geometric. However, we know of no model-theoretic proof of the finer Theorems 1.4 and 1.5. (From a model-theoretic perspective, one may view Theorem 1.5 as an explicit instance of quantifier elimination).

1.4. – Some of the ingredients of our proof are already present in the work of Denef [12, 13], e.g., the use of the weak toroidalisation theorem [4, 3]. We need more ingredients from logarithmic geometry, cf. §5—essentially a few basic properties of log smooth morphisms and log blow-ups. The choice of a log smooth model for the morphism makes some of its arithmetic properties more transparent, and can be seen as a convenient way to come up with a Galois stratification, in the sense of Fried and Sacerdote [14]. On the other hand, we also use work of Serre [31] on Frobenian functions, expanding upon what was done in [25].