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## HÖLDER CONTINUITY OF LYAPUNOV EXPONENT FOR QUASI-PERIODIC JACOBI OPERATORS

BY KAI TAO

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ABSTRACT. — We consider the quasi-periodic Jacobi operator  $H_{x,\omega}$  in  $l^2(\mathbb{Z})$ :  $(H_{x,\omega}\phi)(n) = -b(x + (n+1)\omega)\phi(n+1) - b(x + n\omega)\phi(n-1) + a(x + n\omega)\phi(n) = E\phi(n)$ ,  $n \in \mathbb{Z}$ , where  $a(x)$ ,  $b(x)$  are analytic functions on  $\mathbb{T}$ ,  $b$  is not identically zero, and  $\omega$  obeys some strong Diophantine condition. We consider the corresponding unimodular cocycle. We prove that if the Lyapunov exponent  $L(E)$  of the cocycle is positive for some  $E = E_0$ , then there exist  $\rho_0 = \rho_0(a, b, \omega, E_0)$ ,  $\beta = \beta(a, b, \omega)$  such that  $|L(E) - L(E')| < |E - E'|^\beta$  for any  $E, E' \in (E_0 - \rho_0, E_0 + \rho_0)$ . If  $L(E) > 0$  for all  $E$  in some compact interval  $I$ , then  $L(E)$  is Hölder continuous on  $I$  with Hölder exponent  $\beta = \beta(a, b, \omega, I)$ . In our derivation we follow the refined version of the Goldstein-Schlag method [3] developed by Bourgain and Jitomirskaya [2].

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## 1. Introduction

We consider the quasi-periodic Jacobi operator  $H_{x,\omega}$  in  $l^2(\mathbb{Z})$ :

$$\begin{aligned} (1.1) \quad (H_{x,\omega}\phi)(n) &= -b(x + (n+1)\omega)\phi(n+1) - b(x + n\omega)\phi(n-1) + a(x + n\omega)\phi(n) \\ &= E\phi(n), \quad n \in \mathbb{Z}, \end{aligned}$$

where  $a(x)$ ,  $b(x)$  are real analytic functions on  $\mathbb{T}$ ,  $b$  is not identically zero. Set

$$A(x, E, \omega) = \frac{1}{b(x + \omega)} \begin{pmatrix} a(x) - E & -b(x) \\ b(x + \omega) & 0 \end{pmatrix}.$$

$$\begin{aligned} M_N(x, E, \omega) &= M_{[1,N]}(x, E, \omega) \\ &= A(x + (N-1)\omega, E, \omega)A(x + (N-2)\omega, E, \omega) \cdots A(x, E, \omega). \end{aligned}$$

Define the unimodular matrix

$$\tilde{M}_N(x, E, \omega) = \tilde{M}_{[1,N]}(x, E, \omega) := \frac{M_{[1,N]}(x, E, \omega)}{|\det M_{[1,N]}(x, E, \omega)|^{\frac{1}{2}}}.$$

As

$$\det A(x, E, \omega) = \frac{b(x)}{b(x + \omega)},$$

then

$$(1.2) \quad \det M_{[1,N]}(x, E, \omega) = \prod_{n=0}^{N-1} \frac{b(x + n\omega)}{b(x + (n+1)\omega)} = \frac{b(x)}{b(x + N\omega)},$$

and

$$(1.3) \quad \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| = \log \|M_{[1,N]}(x, E, \omega)\| - \frac{1}{2} \log \left| \frac{b(x)}{b(x + N\omega)} \right|.$$

REMARK 1.1. — (1) Note that

$$\|A(x, E, \omega)\| \leq \frac{C(a, b, E)}{|b(x + \omega)|},$$

where the constant  $C(a, b, E)$  satisfies

$$C(a, b, E_0) = \sup_{|E| \leq E_0} C(a, b, E) < +\infty.$$

Therefore,

$$\frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\| \leq \log C(a, b, E) - \frac{1}{N} \sum_{n=1}^N \log |b(x + n\omega)|.$$

In this paper we always assume that  $|E| \leq E_0$ , where  $E_0$  depends on  $a, b$ . For that matter we suppress  $E$  from the notation of some of the constants involved.

(2)  $\log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \geq 0$ , since  $\tilde{M}_{[1,N]}(x, E, \omega)$  is unimodular.

(3)

$$\begin{aligned} 0 &\leq \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \\ &= \frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\| - \frac{1}{2N} \log \left| \frac{b(x)}{b(x + N\omega)} \right| \\ &\leq \log C(a, b, E) - \frac{1}{N} \sum_{n=1}^N \log |b(x + n\omega)| - \frac{1}{2N} \log \left| \frac{b(x)}{b(x + N\omega)} \right|. \end{aligned}$$

(4) It is a well-known fact that if  $b$  is an analytic function not identically zero, then  $(\log |b|)^2$  is integrable. Set

$$D = \int_{\mathbb{T}} \log |b(\theta)| d\theta.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{T}} \left| \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \right| dx &= \int_{\mathbb{T}} \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| dx \\ &\leq C'(a, b) - D := C''(a, b). \end{aligned}$$

Similarly,

$$\int_{\mathbb{T}} \left( \frac{1}{N} \log \|\tilde{M}_N(x, E, \omega)\| \right)^2 dx \leq \tilde{C}(a, b).$$

(5) Combining (4) with (1.3), we conclude that  $\frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\|$  is integrable, and

$$\frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x, E, \omega)\| dx = \frac{1}{N} \int_{\mathbb{T}} \log \|\tilde{M}_N(x, E, \omega)\| dx.$$

Set

$$L_N(E, \omega) = \frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x, E, \omega)\| dx = \frac{1}{N} \int_{\mathbb{T}} \log \|\tilde{M}_N(x, E, \omega)\| dx.$$

Note that  $L_N(E, \omega) > 0$ . Set

$$B(x, E, \omega) := \begin{pmatrix} a(x) - E & -b(x) \\ b(x + \omega) & 0 \end{pmatrix},$$

and

$$\begin{aligned} T_N(x, E, \omega) &= T_{[1,N]}(x, E, \omega) \\ &:= B(x + (N-1)\omega, E, \omega) B(x + (N-2)\omega, E, \omega) \cdots B(x, E, \omega). \end{aligned}$$

Then

$$\begin{aligned}
 M_{[1,N]}(x, E, \omega) &= T_{[1,N]}(x, E, \omega) \prod_{n=N-1}^0 \frac{1}{b(x + (n+1)\omega)}, \\
 \tilde{M}_{[1,N]}(x, E, \omega) &= \frac{|b(x + N\omega)|^{\frac{1}{2}}}{|b(x)|^{\frac{1}{2}}} M_{[1,N]}(x, E, \omega) \\
 (1.4) \quad &= \frac{|b(x + N\omega)|^{\frac{1}{2}}}{|b(x)|^{\frac{1}{2}}} \prod_{n=0}^{N-1} \frac{1}{b(x + (n+1)\omega)} T_{[1,N]}(x, E, \omega),
 \end{aligned}$$

$$(1.5) \quad \|\tilde{M}_{[1,N]}(x, E, \omega)\| = \prod_{n=0}^{N-1} \frac{1}{|b(x + n\omega)b(x + (n+1)\omega)|^{\frac{1}{2}}} \|T_{[1,N]}(x, E, \omega)\|,$$

and

$$\log \|\tilde{M}_{[1,N]}(x, E, \omega)\| = \log \|T_{[1,N]}(x, E, \omega)\| - \frac{1}{2} \sum_{n=0}^{N-1} \log |b(x + n\omega)b(x + (n+1)\omega)|.$$

Note also for future reference that

$$(1.6) \quad |\det T_{[1,N]}(x, E, \omega)| = \prod_{n=0}^{N-1} |b(x + n\omega)| |b(x + (n+1)\omega)|.$$

Combining (1) with Remark 1.1, we conclude that  $\frac{1}{N} \log \|T_{[1,N]}(x, E, \omega)\|$  is integrable,

$$(1.7) \quad J_N(E, \omega) := \frac{1}{N} \int_{\mathbb{T}} \log \|T_{[1,N]}(x, \omega)\| dx = L_N(E, \omega) + D.$$

Due to the subadditive property, the limits

$$\begin{aligned}
 (1.8) \quad L(E, \omega) &= \lim_{N \rightarrow \infty} \int_{\mathbb{T}} \frac{1}{N} \log \|M_N(x, E, \omega)\| dx = \lim_{N \rightarrow \infty} \int_{\mathbb{T}} \frac{1}{N} \log \|\tilde{M}_N(x, E, \omega)\| dx \\
 &= \lim_{N \rightarrow \infty} L_N(E, \omega),
 \end{aligned}$$

$$(1.9) \quad J(E, \omega) = \lim_{N \rightarrow \infty} \int_{\mathbb{T}} \frac{1}{N} \log \|T_N(x, E, \omega)\| dx = \lim_{N \rightarrow \infty} J_N(E, \omega) = L(E, \omega) + D$$

exist. Moreover,  $L(E, \omega) \geq 0$ . Fix some  $\alpha > 1$ . Throughout this paper we assume that  $\omega \in (0, 1)$  satisfies the Diophantine condition

$$(1.10) \quad \|n\omega\| \geq \frac{C_\omega}{n(\log n)^\alpha} \quad \text{for all } n.$$

It is well known that for a fixed  $\alpha > 1$  almost every  $\omega$  satisfies (1.10).

The main theorem in this paper is