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THE OVERCONVERGENT SITE

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***Abstract*. —** We prove that rigid cohomology can be computed as the cohomology of a site analogous to the crystalline site. Berthelot designed rigid cohomology as a common generalization of crystalline and Monsky-Washnitzer cohomology. Unfortunately, unlike the former, the functoriality of the theory is not built-in. We define the "overconvergent site" which is functorially attached to an algebraic variety. We prove that the category of modules of finite presentation on this ringed site is equivalent to the category of overconvergent isocrystals on the variety. We also prove that their cohomology coincides.

***Résumé* (Le site surconvergent). —** Nous montrons que la cohomologie rigide peut se calculer comme la cohomologie d'un site analogue au site cristallin. Berthelot a conçu la cohomologie rigide comme une généralisation commune de la cohomologie cristalline et de la cohomologie de Monsky-Washnitzer. Malheureusement, contrairement à ce qui se passe en cohomologie cristalline, la fonctorialité de la théorie ne résulte pas directement des définitions. Nous introduisons donc le « site surconvergent » qui est fonctoriellement attaché à une variété algébrique. Nous montrons que la catégorie des modules de présentation finie sur ce site annelé est équivalent à la catégorie des isocristaux surconvergents sur la variété. Nous montrons aussi que leurs cohomologies coïncident.

CONTENTS

CHAPTER 1

INTRODUCTION

In order to give an algebraic description of Betti cohomology, one can use de Rham cohomology which can then be interpreted as the cohomology of the infinitesimal site (see [**21**]). The category of coefficients, locally trivial families of finite dimensional vector spaces, is replaced successively by coherent modules with integrable connections, and then, by modules of finite presentation. In the positive characteristic situation, there is no exact equivalent to Betti cohomology and de Rham cohomology has to be replaced by rigid cohomology (and modules with integrable connections by overconvergent isocrystals). We will define here the overconvergent site which plays in positive characteristic the role that the infinitesimal site plays in characteristic zero. A first hint at this approach is already in Berthelot's fundamental article (see [**10**], 2.3.2.ii) and this is actually the way I liked to define overconvergent isocrystals in my Ph.D. Thesis (see also [**18**], section 1.1 or [**23**], definition 7.1.1). Note that Arthur Ogus introduced the convergent site in [**25**] which is a satisfying solution as long as we are only interested in proper varieties. Of course, for proper smooth varieties, we can also use the crystalline site of Pierre Berthelot [**7**].

Beside its intrinsic interest, there are many reasons to look for such an interpretation of rigid cohomology. For example, we will get for free a Leray spectral sequence giving the overconvergence of the Gauss-Manin connection. Also, our setting should be well-suited to describe Besser's integration (see [**11**]), Chiarellotto-Tsuzuki's descent theory (see [**16**]) or the results of Atsushi Shiho on relative rigid cohomology. Finally, replacing schemes by log-schemes should give a comparison theorem with log-crystalline cohomology. Note that David Brown, who is a student of Bjorn Poonen, makes an essential use of the overconvergent site in his study of rigid cohomology of algebraic stacks (see [**14**]). In order to avoid technical complications, we will not consider étale cohomology nor log-schemes (or algebraic stacks) here.

In our presentation, we will systematically replace rigid geometry with analytic geometry in the sense of Berkovich. I understand that it is unpleasant for those who are accustomed to Tate's theory and feel uncomfortable with Berkovich's. But there are several reasons for this choice. First of all, I really think that classical Tate's theory should be seen as a part of Berkovich theory (using rigid points and Grothendieck topology). Moreover, most young mathematicians start directly with Berkovich's approach. There is also a specific reason here: in the construction of rigid geometry, strict neighborhoods play an essential role; in Berkovich theory those are just usual neighborhoods. Finally, the notion of generic point that is central in Dwork's theory has a very natural interpretation using Berkovich theory (see [**15**]).

Of course, this article owes much to Berthelot's previous work on rigid cohomology (see [**9**], [**10**], [**8**]). We only want to rewrite his theory with a slightly different approach. The reader should note however that we do not make any use of Berthelot's results and that, for this purpose, the article is totally self-contained. In particular, the reader needs not know anything about rigid cohomology. However, since the main results are comparison theorems, I should briefly recall how it works.

Rigid cohomology is a cohomological theory for algebraic varieties over a field k of positive characteristic p with values in vector spaces over a p-adic field K whose residue field is k. The idea is to embed the given variety X into a proper variety Y and then Y into a smooth formal scheme P over the valuation ring $\mathcal{V}$ of K. Then, one considers the limit de Rham cohomology on (strict) neighborhoods of the tube of X inside the generic fiber of P. The hard part in the theory is to show that the cohomological spaces obtained this way are independent of the choices (and that they glue when there exists no embedding as above). There is a relative theory and one may also add coefficients. The coefficients are those of the de Rham theory, namely modules with integrable connections, on a neighborhood of the tube of X, with the extra condition that the connection must be *overconvergent* (overconvergent isocrystals). Here again, this is a local definition and one must show that it does not depend on the choices. It is also important to remark that glueing is the only solution when there is no global embedding as described above. And this is unfortunately the case in general even if it can be avoided in practice.

I want however to emphasize the fact that rigid cohomology was designed as a functorial theory from the beginning and that the purpose of the present article is *not* to fill a possible gap in the original theory. Since this is not completely understood in the mathematical community, it might be necessary to recall how this works. For simplicity, I will only consider the case of absolute cohomology without coefficient of "realizable" varieties (say quasi-projective, if you wish). The general case is more technical but works exactly the same. First of all, rigid cohomology is functorial in triples $X \subset Y \subset P$ made of an open embedding into a variety and a closed embedding into a formal scheme. Thus we can define $\mathrm{H}^i_{\mathrm{rig}}(X \subset Y \subset P)$ and if we are given a morphism $u : P' \to P$ that induces $g : Y' \to Y$ and $f : X' \to X$, we can define $\mathrm{H}^i_{\mathrm{rig}}(u)$. This should be clear. The fundamental theorem of the theory tells us that if f is an