

Astérisque

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Astérisque, tome 209 (1992), p. 99-114

http://www.numdam.org/item?id=AST_1992__209__99_0

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SOME REMARKS ON ELLIPTIC CURVES OVER FUNCTION FIELDS

Tetsuji SHIODA

In my lecture at the Journées Arithmétiques in Geneva (entitled “Mordell-Weil lattices and sphere packings”), I talked on

- 1) a brief survey on lattices and sphere packings,
- 2) basic results on Mordell-Weil lattices, and
- 3) application to sphere packings via supersingular surfaces.

For these topics, the following references are available: 1) [CS,Ch.1], 2) [S3], [S4] and 3) [E], [Oe], [S5].

In this note, instead of reporting on these, I would like to treat some related topics on elliptic curves over a function field, especially some results on the L -function of an elliptic curve over a function field with a finite constant field. Most of them must be known to experts, but the approach based on surface theory and Mordell-Weil lattices seems to provide a natural setting for this subject (cf. [T2],[G],[Mc]). In particular, this method enables one to write down explicit examples of such an L -function in some nontrivial cases.

The contents of this paper are as follows:

1. Elliptic surfaces
2. The L -function of an elliptic curve
3. Supersingular case
4. Rational elliptic surfaces

The present work has been done during my visit to Max-Planck-Institut, Bonn and the University of Geneva. I would like to thank Professor F. Hirzebruch and Professor D. Coray for their kind invitation.

1 Elliptic surfaces

Let us review first some basic results on elliptic surfaces, fixing the notation. Let k be an algebraically closed field of arbitrary characteristic and let K/k be a function field of one variable over k , i.e., $K = k(C)$ for some smooth projective curve C over k . Let E/K be an elliptic curve with a K -rational point O , and let $f : S \rightarrow C$ denote the elliptic surface associated with E/K (the Kodaira-Néron model). The elliptic curve E is recovered from f as its generic fibre and, as is well known, the K -rational points of E can be identified with the sections of f ; for each $P \in E(K)$, (P) denotes the image curve in S of the section $P : C \rightarrow S$. We always assume the condition $(*)$ that f has at least one singular fibre.

Now let $N = \text{NS}(S)$ be the Néron-Severi group of S ; it is a free module of finite rank ρ (=the Picard number of S), which is an (indefinite) integral lattice with respect to the intersection pairing. We denote by T or L the trivial or essential sublattice of N ; by definition, T is the sublattice generated by the zero-section (O) , a fibre and all components of reducible fibres of f , and L is the orthogonal complement of T in N . In particular, we have

$$(1.1) \quad N \otimes \mathbf{Q} = (T \otimes \mathbf{Q}) \oplus (L \otimes \mathbf{Q})$$

and

$$(1.2) \quad \rho = \text{rk } T + \text{rk } L.$$

Further we have

$$(1.3) \quad \text{rk } T = 2 + \sum_{v \in C} (m_v - 1)$$

where m_v is the number of irreducible components of the fibre $f^{-1}(v)$, and $\text{rk } L$ is equal to the Mordell-Weil rank of E/K :

$$(1.4) \quad r := \text{rk } L = \text{rk } E(K).$$

Actually there is a natural isomorphism

$$(1.5) \quad L \otimes \mathbf{Q} \simeq E(K) \otimes \mathbf{Q},$$

which takes the intersection pairing on L to the height pairing on the Mordell-Weil group (up to the sign change); indeed this is essentially how we defined the structure of Mordell-Weil lattices (see [S4]).

Next we consider the cycle map

$$(1.6) \quad \gamma : N \rightarrow H = H^2(S, \mathbf{Q}_l(1))$$

where H stands for the l -adic cohomology group with a fixed prime number $l \neq \text{char}(k)$ (cf. [T1]). It is injective and takes the intersection pairing of N into the cup-product pairing in H . Let us denote by $\text{Trans}(S)$ the orthogonal complement of $\text{Im}(\gamma)$ in H , whose elements are called transcendental cycles on S , and by W the orthogonal complement of $\gamma(T)$ in H . The space W corresponds to what Weil called the *essential part* in the second homology of S (cf. his comments to the paper [1967a] in [W, III]). Then we have

$$(1.7) \quad H \simeq (N \otimes \mathbf{Q}_l) \oplus \text{Trans}(S) \simeq (T \otimes \mathbf{Q}_l) \oplus W$$

and

$$(1.8) \quad W \simeq (L \otimes \mathbf{Q}_l) \oplus \text{Trans}(S)$$

The Lefschetz number of S is defined as

$$(1.9) \quad \lambda := \dim \text{Trans}(S) = b_2 - \rho \quad (b_2 = \dim H^2(S))$$

which is known to be a birational invariant of S .

Proposition 1 *The dimension w of the vector space W is given by*

$$(1.10) \quad w = r + \lambda = b_2 - \text{rk } T.$$

If $\text{char}(k) \neq 2, 3$, then

$$(1.11) \quad w = 4g - 4 + \mu + 2\alpha$$

where g is the genus of C (or of K) and μ (resp. α) is the number of singular fibres of multiplicative (resp. additive) type.

Proof. The first part is immediate from (1.7) and (1.8). The second part is also well-known (cf. [R],[S1]). Let us briefly recall the idea of the proof. From the standard facts in surface theory, we have

$$b_2 = c_2 + 2b_1 - 2 \quad (c_2 = \text{Euler number of } S)$$

where $b_1 = 2g$ since we are assuming the condition (*). On the other hand, we have the following formula for $\text{char}(k) \neq 2, 3$:

$$(1.12) \quad c_2 = \sum_v e_v \quad (e_v = \text{Euler number of } f^{-1}(v))$$

(cf. [K],[Ogg],[Ogu]). Then, by (1.9) and (1.3), we have

$$w = 4g - 4 + \sum_v (e_v - m_v + 1).$$

It remains to check that

$$e_v = m_v \text{ or } m_v + 1$$

according as the fibre $f^{-1}(v)$ is of multiplicative or additive type, which can be done using the classification of singular fibres([K],[N],[T3]). *q.e.d.*

It may be worthwhile to mention the following direct consequence. Simply note that we have $\lambda \geq 0$ in general and $\lambda \geq 2p_g$ (p_g : geometric genus of S) in characteristic 0.

Corollary 2 *If $\text{char}(k) \neq 2, 3$, then*

$$(1.13). \quad r \leq w = 4g - 4 + \mu + 2\alpha.$$

Corollary 3 *Assume $\text{char}(k) = 0$. Then*

$$(1.14) \quad p_g \leq \frac{1}{2}w = 2g - 2 + \frac{1}{2}\mu + \alpha$$

$$(1.15) \quad c_2 = 12(p_g - g + 1) \leq 6(2g - 2 + \mu + 2\alpha)$$

and

$$(1.13'). \quad r \leq 4g - 4 + \mu + 2\alpha - 2p_g.$$

Remark. (a) In case $\text{char}(k) = 2$ or 3 , (1.10) is still valid, but (1.11) should be modified by adding an extra term caused by wild ramifications (cf. [Ogg],[R],[Sa]). In other words, each e_v in (1.12) should be replaced by $e_v + \delta_v$ with a well-defined non-negative integer δ_v so that the right hand side of (1.11) should have the term $\sum_v \delta_v$.

(b) The idea behind equality of expressions in (1.10) and (1.11) was first used by Igusa [I] to define a correct Betti number b_2 of an algebraic surface, and later it was formulated in a more general situation as the so-called Ogg-Shafarevich formula (cf. [R]).

(c) The above (1.14) or its equivalent (1.15) seems to have been proved by many authors again and again, though it was explicitly stated in [S1, Cor.2.7] in 1972. In particular, (1.15) is sometimes called Szpiro's conjecture (cf. [Sz, p.10]); note that we make no assumption of semi-stability ($\alpha = 0$) in the above argument.