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Astérisque, tome 261 (2000), p. 277-333
<http://www.numdam.org/item?id=AST_2000__261__277_0>

Astérisque
261, 2000, p. 277–333

PERIODIC ORBITS, EXTERNALS RAYS AND THE MANDELBROT SET: AN EXPOSITORY ACCOUNT

by

John Milnor

Dedicated to Adrien Douady on the occasion of his sixtieth birthday

Abstract. — A presentation of some fundamental results from the Douady-Hubbard theory of the Mandelbrot set, based on the idea of "orbit portrait": the pattern of external rays landing on a periodic orbit for a quadratic polynomial map.

1. Introduction

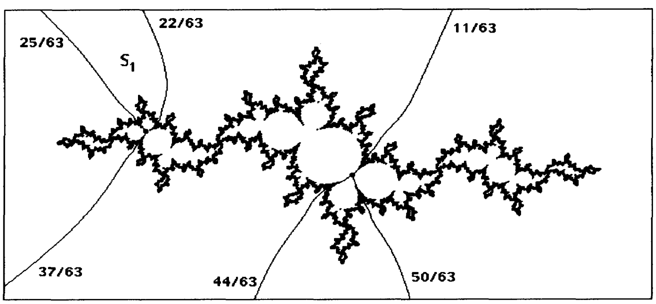

FIGURE 1. Julia set for $z \mapsto z^2 + (\frac{1}{4}e^{2\pi i/3} - 1)$ showing the six rays landing on a period two parabolic orbit. The associated orbit portrait has characteristic arc $\mathcal{I} = (22/63, 25/63)$ and valence $v = 3$ rays per orbit point.

A key point in Douady and Hubbard's study of the Mandelbrot set M is the theorem that every parabolic point $c \neq 1/4$ in M is the landing point for exactly two external rays with angles which are periodic under doubling. (See **[DH2]**. By

1991 ***Mathematics Subject Classification.*** — 30D05.
Key words and phrases. — Mandelbrot set, periodic orbit.

definition, a parameter point is *parabolic* if and only if the corresponding quadratic map has a periodic orbit with some root of unity as multiplier.) This note will try to provide a proof of this result and some of its consequences which relies as much as possible on elementary combinatorics, rather than on more difficult analysis. It was inspired by §2 of the recent thesis of Schleicher **[S1]**, which contains very substantial simplifications of the Douady-Hubbard proofs with a much more compact argument, and is highly recommended. (See also **[S2]**, **[LS]**.) The proofs given here are rather different from those of Schleicher, and are based on a combinatorial study of the angles of external rays for the Julia set which land on periodic orbits. (Compare **[A]**, **[GM]**.) As in **[DH1]**, the basic idea is to find properties of M by a careful study of the dynamics for parameter values outside of M. The results in this paper are mostly well known; there is a particularly strong overlap with **[DH2]**. The only claim to originality is in emphasis, and the organization of the proofs. (Similar methods can be used for higher degree polynomials with only one critical point. Compare **[S3]**, **[E]**, and see **[PR]** for a different approach. For a theory of polynomial maps which may have many critical points, see **[K]**.)

We will assume some familiarity with the classical Fatou-Julia theory, as described for example in **[Be]**, **[CG]**, **[St]**, or **[M2]**.

Standard Definitions. — (Compare Appendix A.) Let $K = K(f_c)$ be the *filled Julia set*, that is the union of all bounded orbits, for the quadratic map

$$f(z) = f_c(z) = z^2 + c.$$

Here both the parameter c and the dynamic variable z range over the complex numbers. The *Mandelbrot set* M can be defined as the compact subset of the *parameter plane* (or c-plane) consisting of all complex numbers c for which $K(f_c)$ is connected. We can also identify the complex number c with one particular point in the *dynamic plane* (or z-plane), namely the *critical value* $f_c(0) = c$ for the map f_c. The parameter c belongs to M if and only if the orbit $f_c : 0 \mapsto c \mapsto c^2 + c \mapsto \cdots$ is bounded, or in other words if and only if $0, c \in K(f_c)$. Associated with each of the compact sets $K = K(f_c)$ in the dynamic plane there is a *potential function* or *Green's function* $G^K : \mathbb{C} \to [0, \infty)$ which vanishes precisely on K, is harmonic off K, and is asymptotic to $\log|z|$ near infinity. The family of *external rays* of K can be described as the orthogonal trajectories of the level curves $G^K =$ constant. Each such ray which extends to infinity can be specified by its angle at infinity $t \in \mathbb{R}/\mathbb{Z}$, and will be denoted by $\mathcal{R}_t^K$. Here c may be either in or outside of the Mandelbrot set. Similarly, we can consider the potential function G^M and the external rays $\mathcal{R}_t^M$ associated with the Mandelbrot set. We will use the term *dynamic ray* (or briefly K-*ray*) for an external ray of the filled Julia set, and *parameter ray* (or briefly M-*ray*) for an external ray of the Mandelbrot set. (Compare **[S1]**, **[S2]**.)

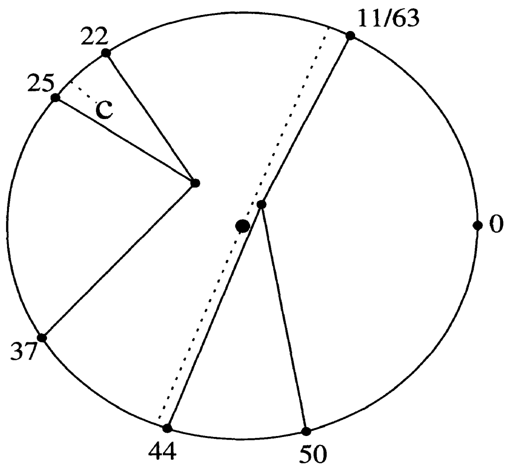

FIGURE 2. Schematic diagram illustrating the orbit portrait (1).

Definition. — Let $\mathcal{O} = \{z_1, \ldots, z_p\}$ be a periodic orbit for f. Suppose that there is some rational angle $t \in \mathbb{Q}/\mathbb{Z}$ so that the dynamic ray $\mathcal{R}_t^{K(f)}$ lands at a point of $\mathcal{O}$. Then for each $z_i \in \mathcal{O}$ the collection A_i consisting of all angles of dynamic rays which land at the point z_i is a finite and non-vacuous subset of $\mathbb{Q}/\mathbb{Z}$. The collection $\{A_1, \ldots, A_p\}$ will be called the *orbit portrait* $\mathcal{P} = \mathcal{P}(\mathcal{O})$. As an example, Figure 1 shows a quadratic Julia set having a parabolic orbit with portrait

$$\mathcal{P} = \{\ \{22/63,\ 25/63,\ 37/63\},\ \ \{11/63,\ 44/63,\ 50/63\}\ \}. \tag{1}$$

It is often convenient to represent such a portrait by a schematic diagram, as shown in Figure 2. (For details, and an abstract characterization of orbit portraits, see §2.)

The number of elements in each A_i (or in other words the number of K-rays which land on each orbit point) will be called the *valence* v. Let us assume that $v \geq 2$. Then the v rays landing at z cut the dynamic plane up into v open regions which will be called the *sectors* based at the orbit point $z \in \mathcal{O}$. The *angular width* of a sector S will mean the length of the open arc I_S consisting of all angles $t \in \mathbb{R}/\mathbb{Z}$ with $\mathcal{R}_t^K \subset S$. (We use the word 'arc' to emphasize that we will identify $\mathbb{R}/\mathbb{Z}$ with the 'circle at infinity' surrounding the plane of complex numbers.) Thus the sum of the angular widths of the v distinct sectors based at an orbit point z is always equal to $+1$. The following result will be proved in 2.11.

***Theorem 1.1* (The Critical Value Sector S_1).** — *Let $\mathcal{O}$ be an orbit of period $p \geq 1$ for $f = f_c$. If there are $v \geq 2$ dynamic rays landing at each point of $\mathcal{O}$, then there is one and only one sector S_1 based at some point $z_1 \in \mathcal{O}$ which contains the critical value $c = f(0)$, and whose closure contains no point other than z_1 of the orbit $\mathcal{O}$. This* *critical value sector S_1 can be characterized, among all of the pv sectors based at the various points of $\mathcal{O}$, as the unique sector of smallest angular width.*

It should be emphasized that this description is correct whether the filled Julia set K is connected or not.

Our main theorem can be stated as follows. Suppose that there exists some polynomial f_{c_0} which admits an orbit $\mathcal{O}$ with portrait $\mathcal{P}$, again having valence $v \geq 2$. Let $0 < t_- < t_+ < 1$ be the angles of the two dynamic rays $\mathcal{R}^K_{t_\pm}$ which bound the critical value sector S_1 for f_{c_0}.

Theorem 1.2 **(The Wake $W_\mathcal{P}$).** — *The two corresponding parameter rays $\mathcal{R}^M_{t_\pm}$ land at a single point $\mathbf{r}_\mathcal{P}$ of the parameter plane. These rays, together with their landing point, cut the plane into two open subsets $W_\mathcal{P}$ and $\mathbb{C} \smallsetminus \overline{W}_\mathcal{P}$ with the following property: A quadratic map f_c has a repelling orbit with portrait $\mathcal{P}$ if and only if $c \in W_\mathcal{P}$, and has a parabolic orbit with portrait $\mathcal{P}$ if and only if $c = \mathbf{r}_\mathcal{P}$.*

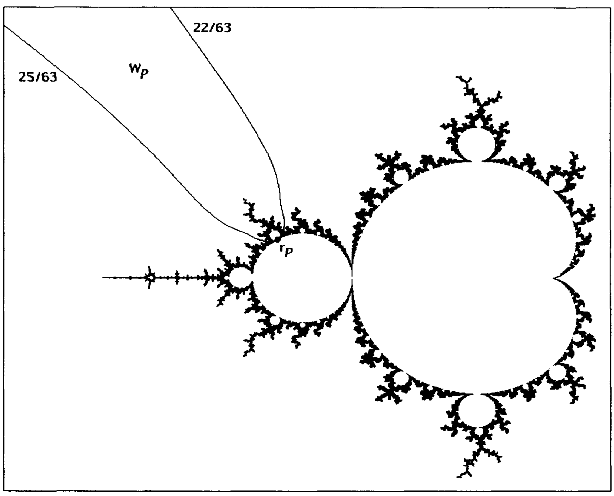

FIGURE 3. The boundary of the Mandelbrot set, showing the wake $W_\mathcal{P}$ and the root point $\mathbf{r}_\mathcal{P} = \frac{1}{4}\, e^{2\pi i/3} - 1$ associated with the orbit portrait of Figure 1, with characteristic arc $\mathcal{I}_\mathcal{P} = (22/63, 25/63)$.

In fact this will follow by combining the assertions 3.1, 4.4, 4.8, and 5.4 below.

Definitions. — This open set $W_\mathcal{P}$ will be called the $\mathcal{P}$*-wake* in parameter space (compare Atela **[A]**), and $\mathbf{r}_\mathcal{P}$ will be called the *root point* of this wake. The intersection $M_\mathcal{P} = M \cap \overline{W}_\mathcal{P}$ will be called the $\mathcal{P}$*-limb* of the Mandelbrot set. The open arc $I_{S_1} = (t_-, t_+)$ consisting of all angles of dynamic rays $\mathcal{R}^K_t$ which are contained in the interior of S_1, or all angles of parameter rays $\mathcal{R}^M_t$ which are contained in $W_\mathcal{P}$, will be called the *characteristic arc* $\mathcal{I} = \mathcal{I}_\mathcal{P}$ for the orbit portrait $\mathcal{P}$. (Compare 2.6.)