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INTEGRAL FORMULAS FOR TWO-LAYER SCHUR AND WHITTAKER PROCESSES

by Guillaume Barraquand

Abstract. — Stationary measures of last passage percolation with geometric weights and the log-gamma polymer in a strip of the $\mathbb{Z}^2$ lattice are characterized in [7] using variants of Schur and Whittaker processes, called two-layer Gibbs measures. In this article, we prove contour integral formulas characterizing the multipoint joint distribution of two-layer Schur and Whittaker processes. We also express them as Doob transformed Markov processes with explicit transition kernels. As an example of the application of our formulas, we compute the growth rate of the KPZ equation on $[0, L]$ with arbitrary boundary parameters.

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Résumé (*Formules intégrales pour les processus de Schur et Whittaker à deux couches*). — Les mesures stationnaires du modèle de percolation de dernier passage avec poids géométriques, ainsi que celles du modèle de polymère dirigé avec poids log-gamma, sont caractérisées dans [7] au moyen de variantes de processus de Schur ou de Whittaker, appelées mesures de Gibbs à deux couches. Dans cet article, nous établissons des formules, sous forme d'intégrales de contour, qui caractérisent la loi jointe en plusieurs points de ces processus de Schur ou de Whittaker à deux couches. Nous les exprimons aussi comme la transformation de Doob d'un processus de Markov dont le noyau de transition est donné explicitement. À titre d'exemple d'application de nos formules, nous calculons le taux de croissance de l'équation KPZ sur l'intervalle $[0, L]$ pour des valeurs arbitraires de paramètres de bords.

1. Introduction

1.1. Preface. — Integrable random systems defined on unbounded domains often admit particularly simple stationary measures, described by families of independent and identically distributed random variables. In finite volume with open boundary conditions, however, stationary measures are typically more complicated, with spatial correlations and a non-trivial dependence on boundary conditions. The archetype example is the asymmetric simple exclusion process on ℓ sites, connected to boundary reservoirs (open ASEP). Computing its stationary measure boils down to finding the principal eigenvector of a rather simple Markov matrix, indexed by elements of the state space $\{0,1\}^\ell$. But this is not easy. In 1993, [31] introduced an elegant method, called the matrix product ansatz, to determine the stationary measure of open ASEP. This method is remarkably powerful and has been extended in various directions over the years. Yet, the matrix product ansatz becomes increasingly difficult to apply on models with a larger state space, for instance a state space such as $\mathbb{Z}^\ell$ or $\mathbb{R}^\ell$.

In order to characterize stationary measures of geometric last passage percolation and the log-gamma polymer in a strip of the $\mathbb{Z}^2$ lattice (i.e., paths are constrained to stay between two walls), another method was introduced in [7]. It relies on connections between these models and families of symmetric functions, namely the Schur functions for last passage percolation and Whittaker functions for the log-gamma polymer model.

Stationary measures of out of equilibrium systems in finite domains are typically not expected to be Gibbs measures. Yet the method of [7] expresses them as marginals of a Gibbs measure on a larger graph. The structure of this graph and the expression of the Boltzmann weights are closely related to the branching rule satisfied by the corresponding family of symmetric functions. These measures, called two-layer Gibbs measures in [7], can also be seen as variants of Schur or Whittaker processes [9, 10, 11, 27, 47]. While the matrix product ansatz characterizes stationary measures through formulas, two-layer

Gibbs measures yield directly a probabilistic description, which is suitable to taking scaling limits, notably to the Kardar–Parisi–Zhang (KPZ) equation. The goal of the present paper is to complement this qualitative and probabilistic description with more quantitative information.

Our results are very similar for the two models: geometric last passage percolation and the log-gamma polymer in a strip (defined in Section 1.2).

1. We prove explicit contour integral formulas for the multipoint Laplace transform of the stationary process. Our results have a similar form as formulas for the stationary measures of open ASEP [12] and the open KPZ equation [26].
2. We provide alternative descriptions of the two-layer Gibbs measures introduced in [7] in terms of explicit Markov processes with a well-chosen initial condition. These results may be seen as discrete analogues of [16] (they also have similarities with [30]).

In Section 4, we describe an application of our exact formulas to the KPZ equation. Assuming the convergence of the log-gamma polymer free energy on a strip to the open KPZ equation (a precise statement is conjectured in [7]), we obtain an explicit expression for the growth rate of the solution. Starting from stationary initial condition, the solution $h(t,x)$ of the KPZ equation on $[0,L]$ with boundary parameters $u,v\in\mathbb{R}$ is such that

$$\mathbb{E}\left[h(t,0)\right] = t\left(\frac{-1}{24}+\frac{1}{2}\partial_L \log \mathcal{Z}_{u,v}(L)\right),$$

where

$$\mathcal{Z}_{u,v}(L) = \int_{\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi}\left|\frac{\Gamma(u+z)\Gamma(v+z)}{\Gamma(2z)}\right|^2 \frac{e^{z^2L}}{2}.$$

The formulas above hold only when $u,v>0$, but they may be analytically continued to study other values of u or v. In the large L limit, we obtain a phase transition with three phases (Corollary 4.3). The phase diagram is, of course, the same as that of open ASEP [31], as predicted by universality.

As an application of the Markovian description, we discuss limits of stationary measures on a finite domain as the size of the domain goes to infinity. As shown in [15] in the context of the open KPZ equation, a Markovian description is particularly convenient for such task. We prove that two-layer Schur and Whittaker processes can be seen as Markov processes, more precisely Doob transforms of killed random walks. We establish exact formulas for the transition kernels and explain how they can be used to study the convergence of stationary measures on a strip as the width of the strip goes to infinity.

The method of [7] is expected to apply also to other models related to other families of symmetric functions. While we restrict ourselves here to the cases studied in [7], we believe that the methods used in the present paper could be adapted to obtain integral formulas for two-layer Gibbs measures related to