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SIMILARITY SURFACES, CONNECTIONS AND THE MEASURABLE RIEMANN MAPPING THEOREM

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ABSTRACT. — This article studies a particular process that approximates solutions of the Beltrami equation (straightening of ellipse fields, a.k.a. measurable Riemann mapping theorem) on $\mathbb{C}$. Introducing a sequence of similarity surfaces constructed by gluing polygons, we explain the relation between their conformal uniformization and the Schwarz–Christoffel formula. Numerical aspects, in particular the efficiency of the process, are not studied, but we draw interesting theoretical consequences. First, we give an independent proof of the analytic dependence, on the data (the Beltrami form), of the solution of the Beltrami equation (Ahlfors–Bers theorem). For this we prove, without using the Ahlfors–Bers theorem, the holomorphic dependence, with respect to the polygons, of the Christoffel symbol appearing in the Schwarz–Christoffel formula. Second, these Christoffel symbols define a sequence of parallel transports on the range, and in the case of a data that is C^2 with compact support, we prove that it converges to the parallel transport associated to a particular affine connection, which we identify.

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Résumé (*Surfaces de similitude, connexions et théorème de l'application conforme mesurable*). — Dans cet article, nous étudions un schéma d'approximation des solutions de l'équation de Beltrami, c'est-à-dire le redressement des champs d'ellipses, également connu sous le nom de théorème de l'application conforme mesurable sur $\mathbb{C}$. En introduisant une suite de surfaces de similitude obtenues par recollement de polygones, nous expliquons la relation entre leur uniformisation conforme et la formule de Schwarz-Christoffel. Les aspects numériques, en particulier l'efficacité de l'algorithme, ne sont pas étudiés ici mais nous donnons un certain nombre de conséquences théoriques intéressantes. D'abord, nous donnons une preuve indépendante que la solution de l'équation de Beltrami dépend analytiquement des données initiales, c'est-a-dire de la forme de Beltrami (théorème d'Ahlfors-Bers). Pour cela, nous prouvons, sans utiliser le théorème de Ahlfors-Bers, que les symboles de Christoffel apparaissant dans la formule de Schwarz-Christoffel dépendent holomorphiquement des polygones. Ensuite, ces symboles de Christoffel définissent une suite de transports parallèles sur le domaine. Dans le cas où la forme de Beltrami est C^2 et à support compact, nous prouvons que cette suite converge vers le transport parallèle associé à une certaine connexion affine que nous identifions.

Introduction

More than 50 years later, Chapter II of [3] remains an excellent introduction to the theory of quasiconformal mappings, and we highly recommend it.[1] This book also includes a proof of the measurable Riemann mapping theorem in Chapter V (see Section 7 here for a statement). This proof makes use of integral operators with singular kernels (Ahlfors–Beurling operator), L^p spaces for $p > 2$, letting $p \longrightarrow 2$, and an inequality due to Calderón and Zygmund. There have been prior and posterior proofs with different approaches in special cases or in the general case, depending on the data μ in the statement (the article [19] gives a short historical overview and references):

- Gauss (1825) for an $\mathbb{R}$-analytic μ, by complexifying $\mathbb{R}^2$ into $\mathbb{C}^2$ and using a clever trick that does not extend to the C^∞ class;
- Korn (1914) and Lichtenstein (1916) for Hölder-continuous μ; Lichtenstein's method already involves integral operators. Korn uses a fixed point method involving solving the Laplacian;
- Lavrentiev (1935) for a continuous μ, by constructing approximations to the solution using conformal geometric methods (we sum up the method in Part II) [27, 28];
- Morrey (1936) for a general, measurable μ, using a density argument to reduce to solving the case where μ is analytic, [33];

1. Or the second edition [4], which was issued in 2006.

to cite only a few. According to [13] (translated in [12]), the idea to introduce the Ahlfors–Beurling operator (it did not bear that name at that time) for this problem is due to Vekua [49]. Concerning the holomorphic dependence on the data μ, all the proofs that the authors of the present article know use the Ahlfors–Beurling operator. These works include:

- Ahlfors and Bers, via L^p spaces.
- Buff–Douady, via L^2 spaces only and the Fourier transformation.
- Glutsyuk, via a limit of a version for the torus, which is proved using Fourier series.

Part II in the present article is an addition to this list, with the difference that we do not use the Ahlfors–Beurling operator. Our proof uses distributions and L^2 spaces for the definition of quasiconformal maps, a classical compactness argument via an L^2 estimate, a new ingredient that is similarity surfaces, and the Poincaré–Koebe theorem.[2]. It bears resemblance with the approach of Lavrentiev, and, in fact, the article of Lavrentiev avoids the use of the Poincaré–Koebe theorem, so it is not impossible that the proof we present may be adapted too to avoid this use, but this does not seem obvious.

Structure and content of the article. — Part I is an introduction to the subject of similarity surfaces with a focus on ones that are conformally equivalent to punctured Riemann spheres and obtained by gluing finitely many convex polygons, possibly unbounded (Section 2). In Section 3,we draw a link with a generalization of the Schwarz–Christoffel formula: it expresses similarity charts. We believe that this fact was known before but we do not have references for this. In the formula, a particular rational map appears, which is the Christoffel symbol (expression in a chart) of a particular conformal connection. Section 4 is devoted to a proof of holomorphic dependence of the rational map appearing as a Christoffel symbol, when the polygons that are glued are modified by letting the vertices vary holomorphically. A key point is to completely avoid using the measurable Riemann mapping theorem in this proof. Such an insistence induces complications that are dealt with in Section 5.4.

Part II states the measurable Riemann mapping theorem (solution of the Beltrami equation associated to a Beltrami form μ) and details our proof. A crucial point is the holomorphic dependence, which relies on the holomorphic dependence of the Christoffel symbol proved in Part I. We first describe an (already known) density method with some amount of generality. Then we apply the results of the previous section to finalize the proof.

The density method does not involve the Ahlfors–Beurling operator nor L^p spaces beyond L^1, L^2. It involves the notion of distribution and the Sobolev

2. … Of which there are several proofs that do not use Beltrami forms. See, for instance, Chapter 10 of [5], or [22].